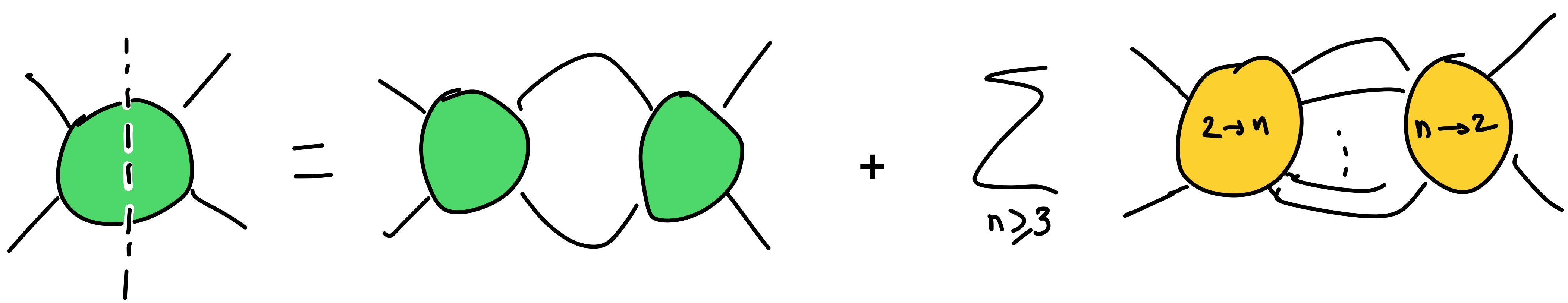
Scattering from production in 2d

Piotr Tourkine,
LPTHE/CNRS, Paris & CERN

ITMP online seminar series
03/02/2021



based on: [arXiv:2101.05211]
P. Tourkine, A. Zhiboedov

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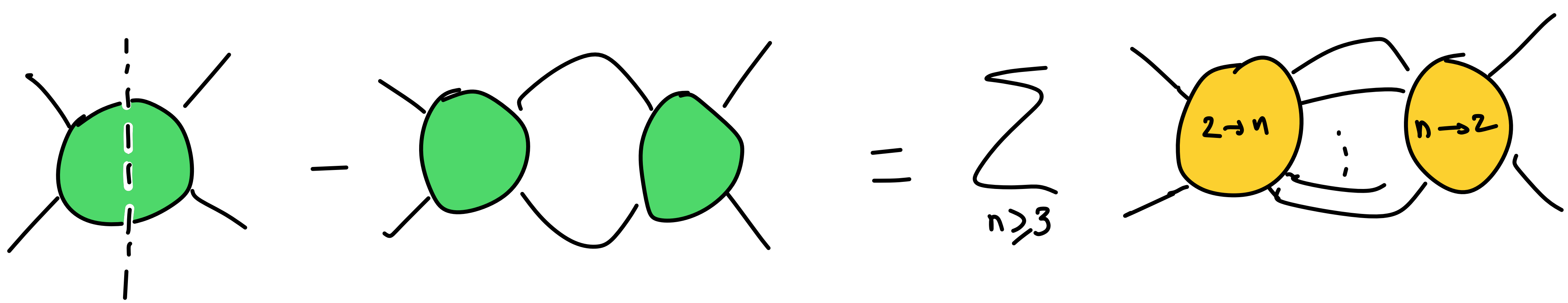


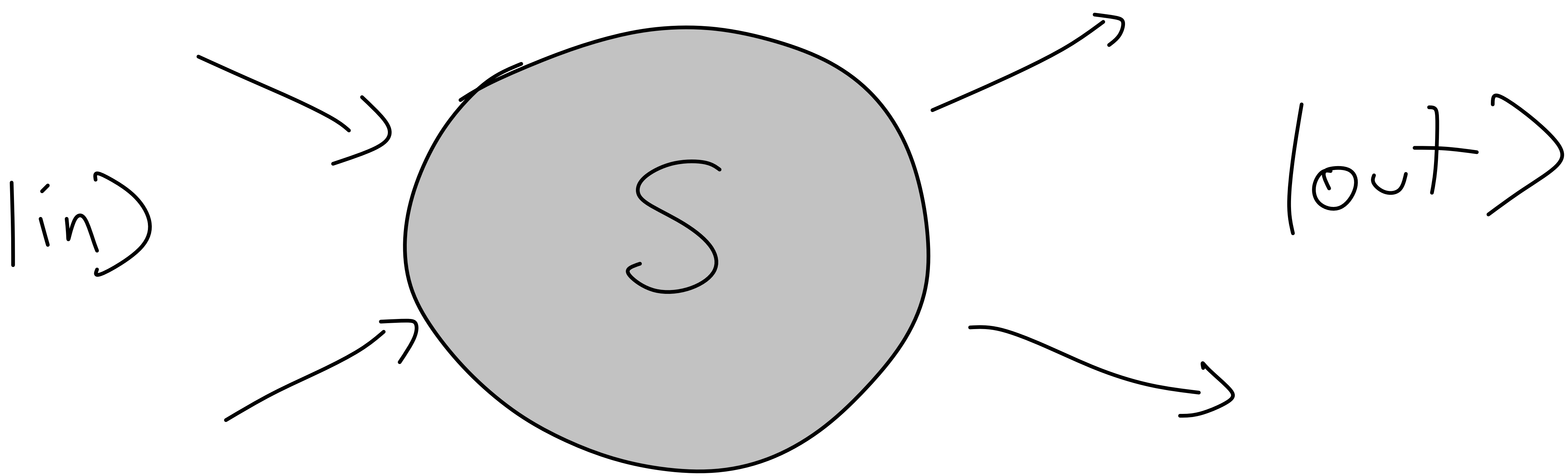
based on: [arXiv:2101.05211]
P. Tourkine, A. Zhiboedov

Plan

- General intro
  - motivations & presentation of the problem
- Results
  - Fixed-point iteration
  - Newton's method
- Discussion

The S-matrix

- Most basic observable of QFT



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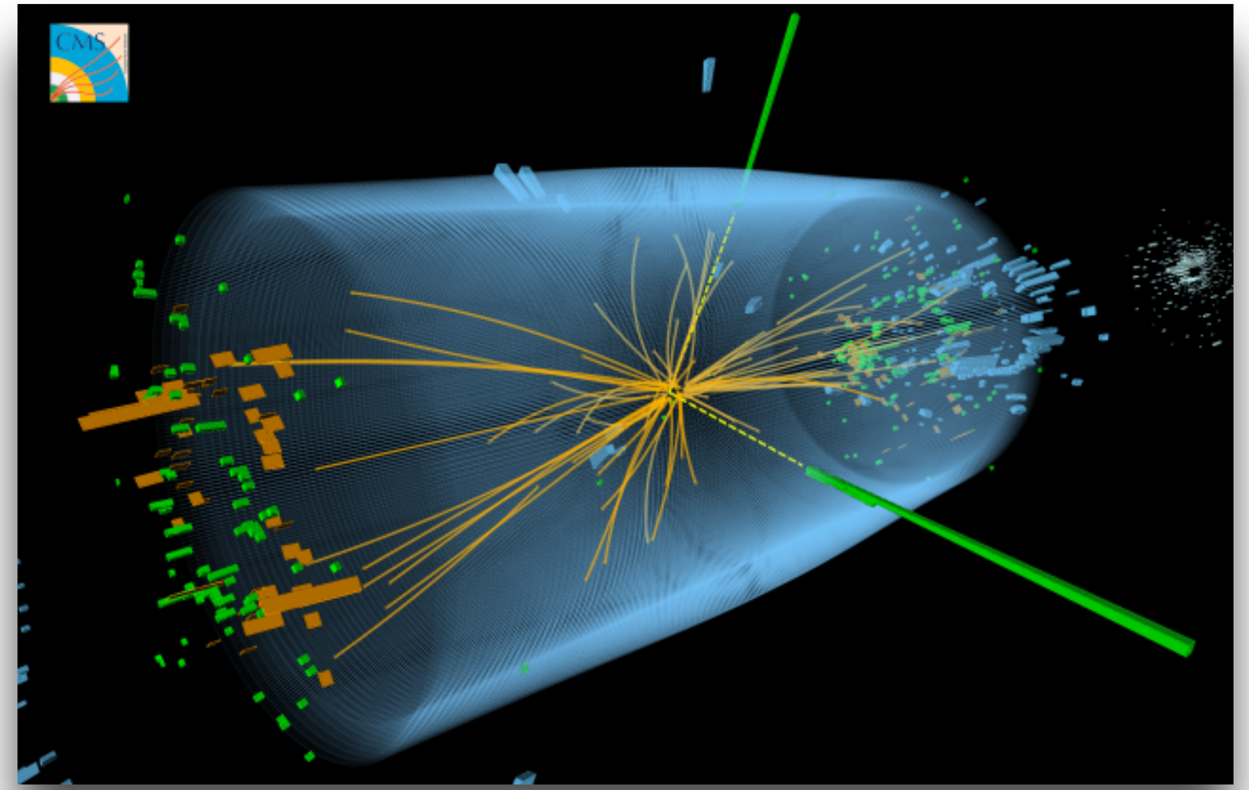
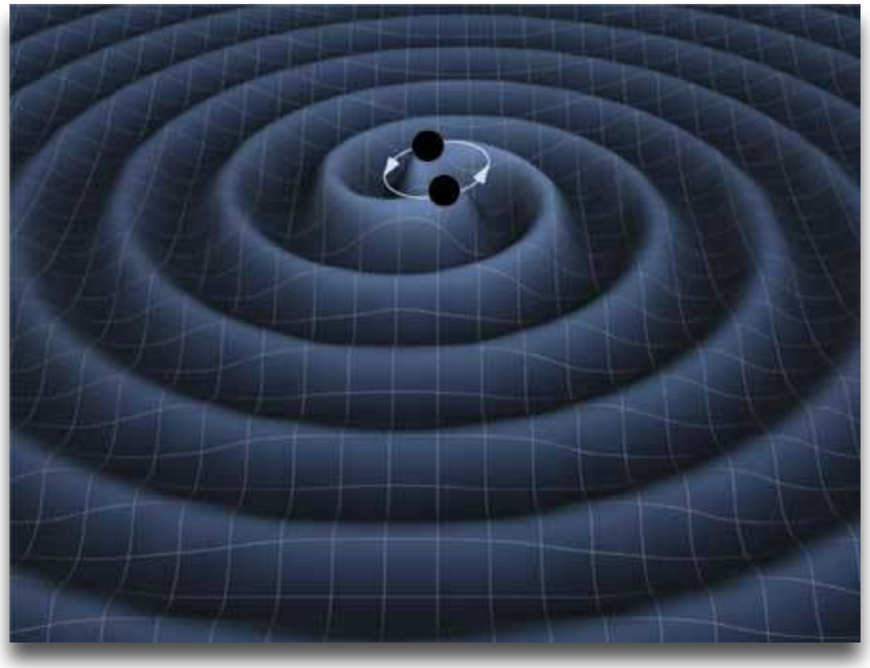
The S-matrix

- Weakly coupled theories: direct approach, perturbative methods, Feynman rules





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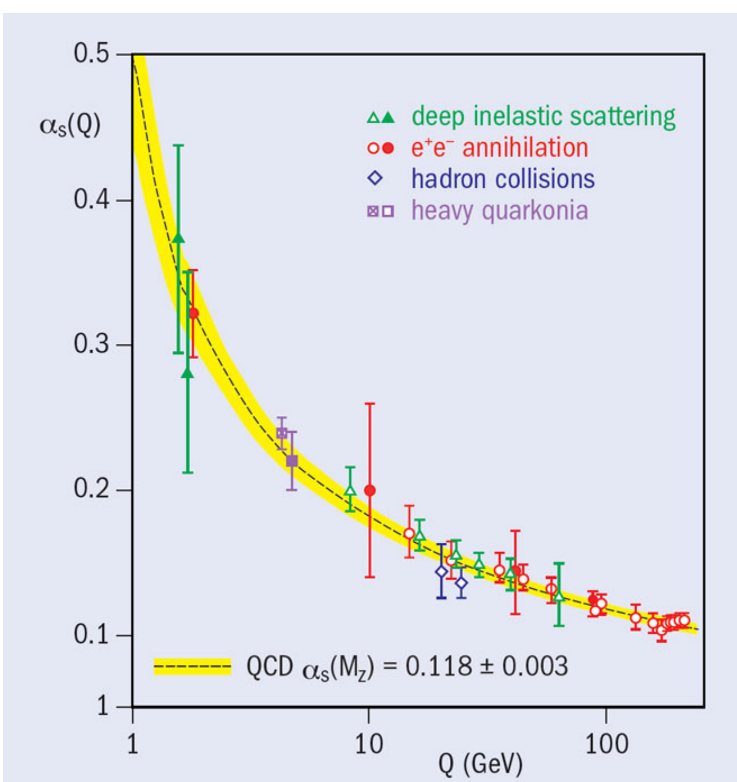
The S-matrix

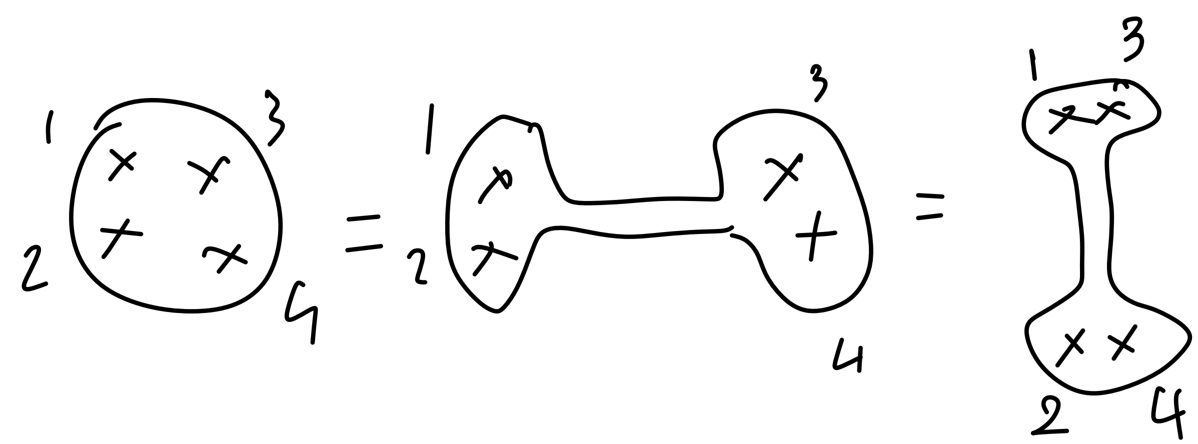
- Full non-perturbative approach: bootstrap. Determines full S-matrix from a set of consistent axioms. “The bootstrap”



The CFT bootstrap

- Revided in CFT's

[1] [arXiv:1203.6064] Phys.Rev. **D86** (2012) 025022
Solving the 3D Ising Model with the Conformal Bootstrap
S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, A. Vichi



- Solve crossing (linear)

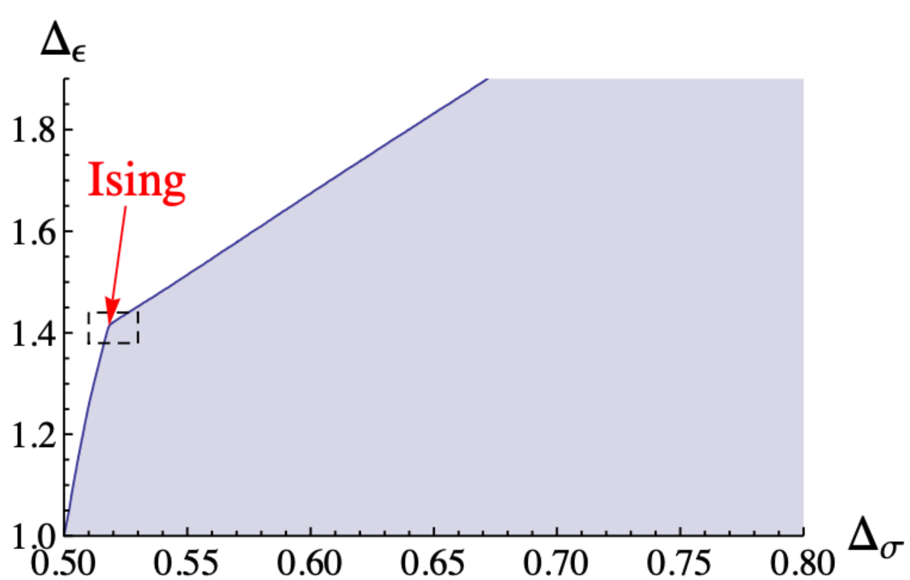


fig 3 of [1]

crossing equation in CFTs

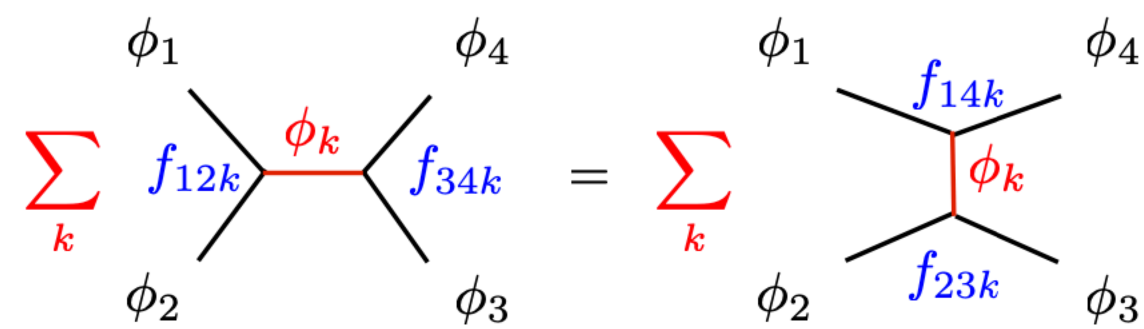


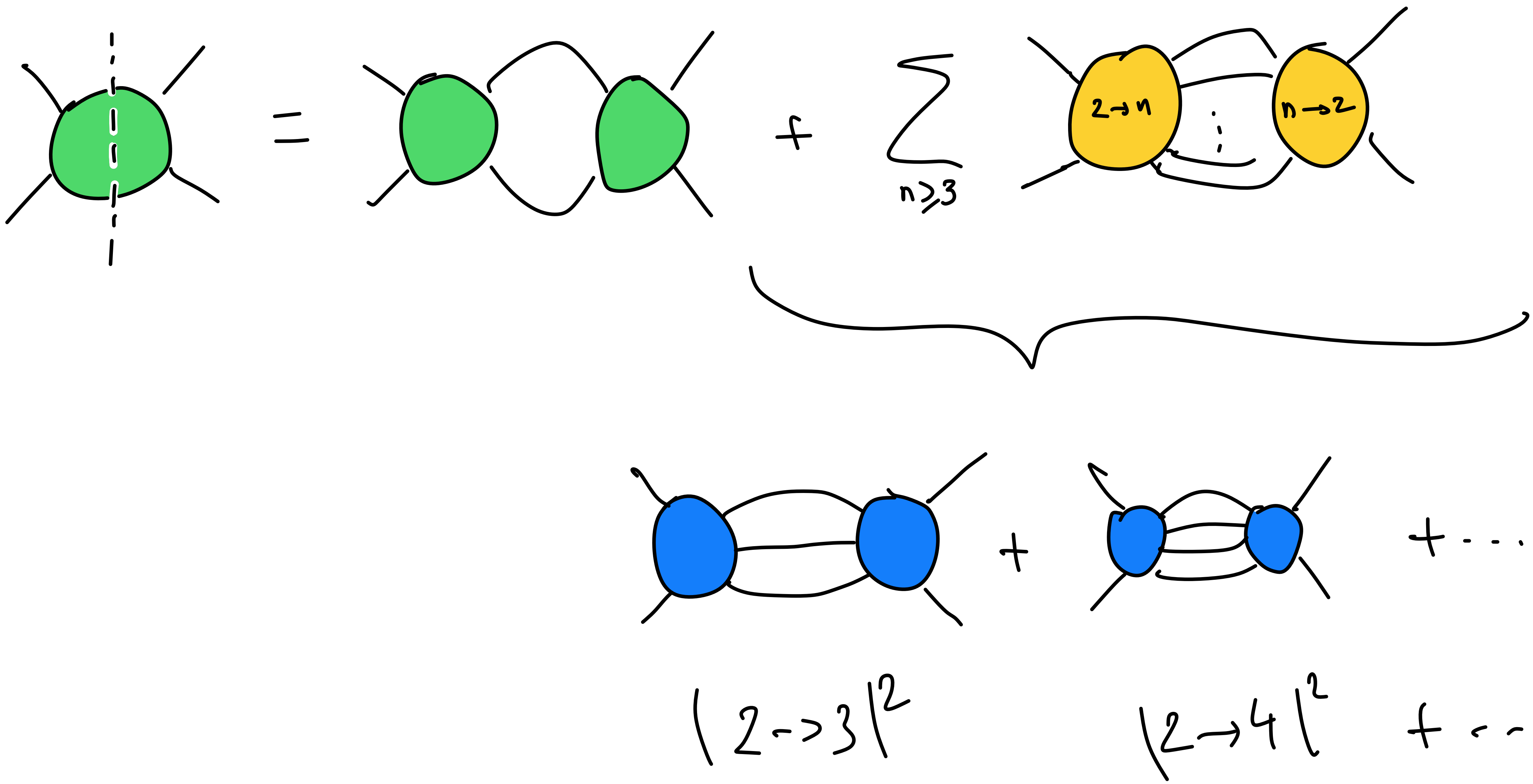
fig 1 of [1]

for S-matrix

- Crossing (linear) + Unitarity (non-linear)
- Impressive results since the 50s’-60s’
- Today, numerical techniques bootstrap are being re-applied to the S-matrix

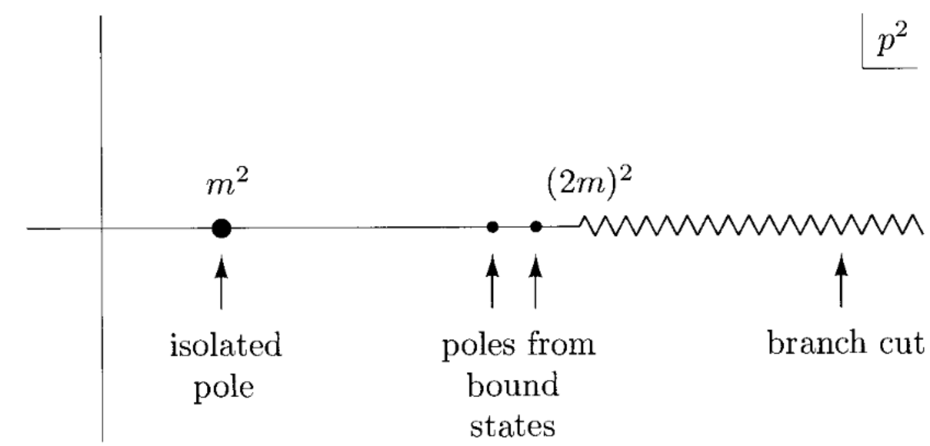
S-matrix unitarity

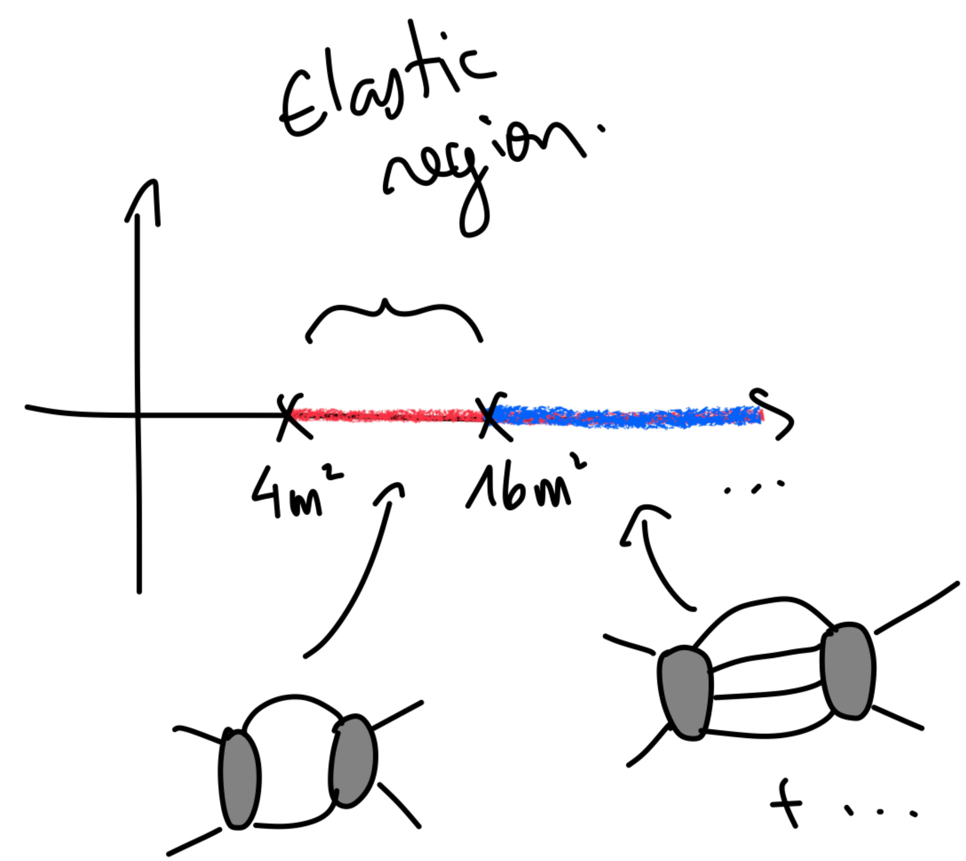
- $S^\dagger S = 1$
- $S = 1 + iT \implies 2i\Im T_{ab} = T^\dagger_{ac} T_{cb}$
- Sum over c: sum over complete set of states;
- $\sum_{|c\rangle} = \sum_{2-pt\ states} \int_{phase-space} + \sum_{3-pt\ states} \int + \dots$
- For 2 to 2, we have $\Im T_{2\to 2} = \sum_{n=2}^{\infty} T_{2\to n} T^*_{n\to 2}$
- Where $\Im T(s) = (T(s+i\epsilon) - T(s-i\epsilon)/(2i) = \mathrm{Disc}_s T(s)/(2i)$


=
+
n≥3
2→n
n→2
+
+ ...
|2->3|²
|2→4|²
+ ...

Our set-up

- We consider the 2-to-2 scattering of lightest states in a gapped QFT
- Goal: construct functions that satisfy the following S-matrix axioms: unitarity, crossing and Mandelstam analyticity
- No such function was built in d>2 as of today
- In 4 dimensions, *given crossing*, one property is particularly difficult to enforce: *Elastic unitarity*

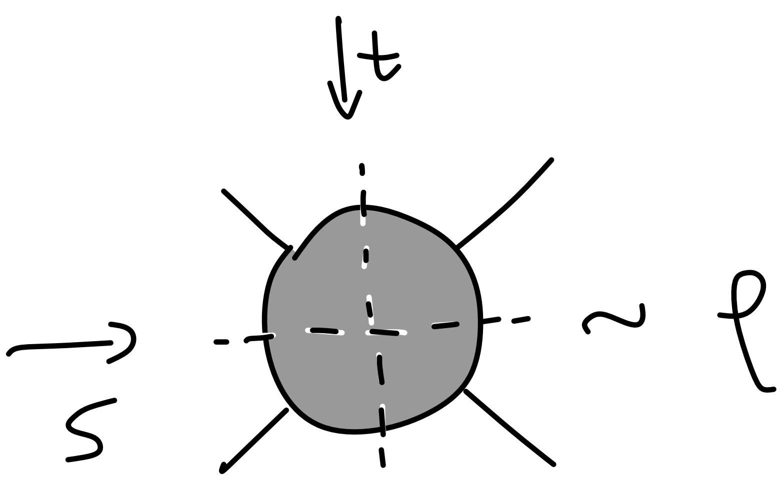




Elastic unitarity in 4d

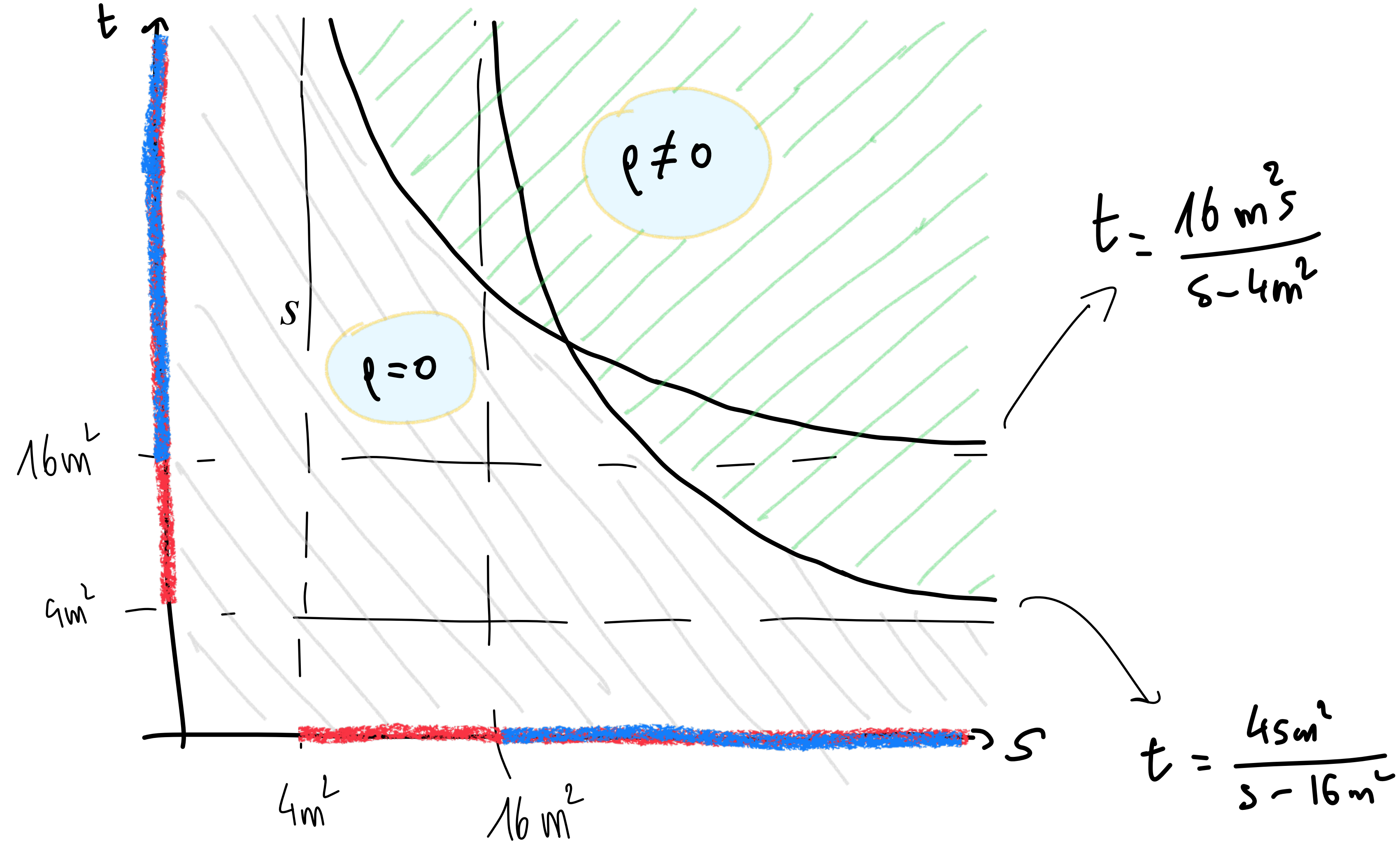
Correreira, Sever, Zhiboedov ‘20

for now,
$\rho \sim$ double disc:



$\rho \sim \mathrm{disc}_t \mathrm{disc}_s T(s,t)$

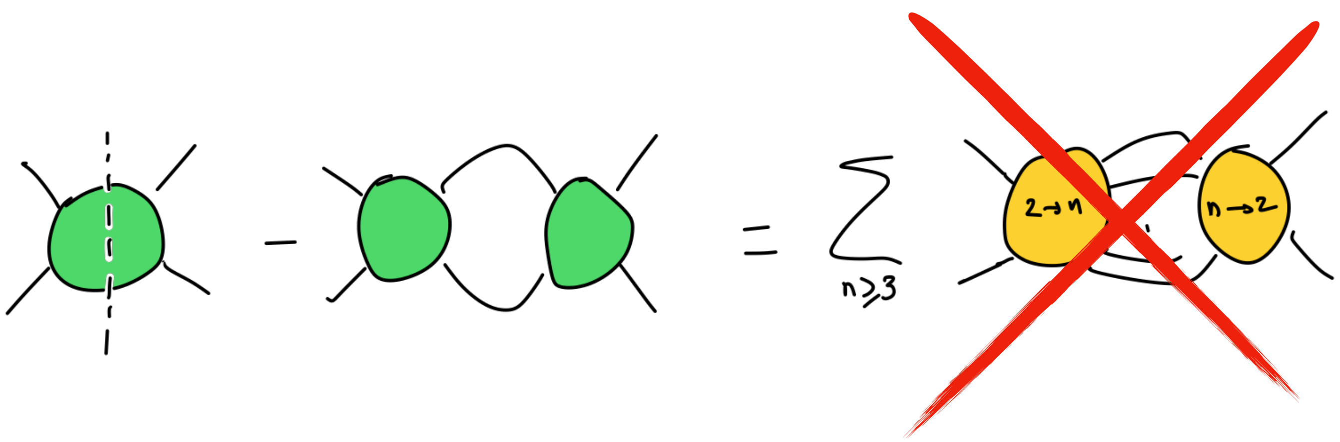
s ~ center of mass energy
t ~ momentum transfer



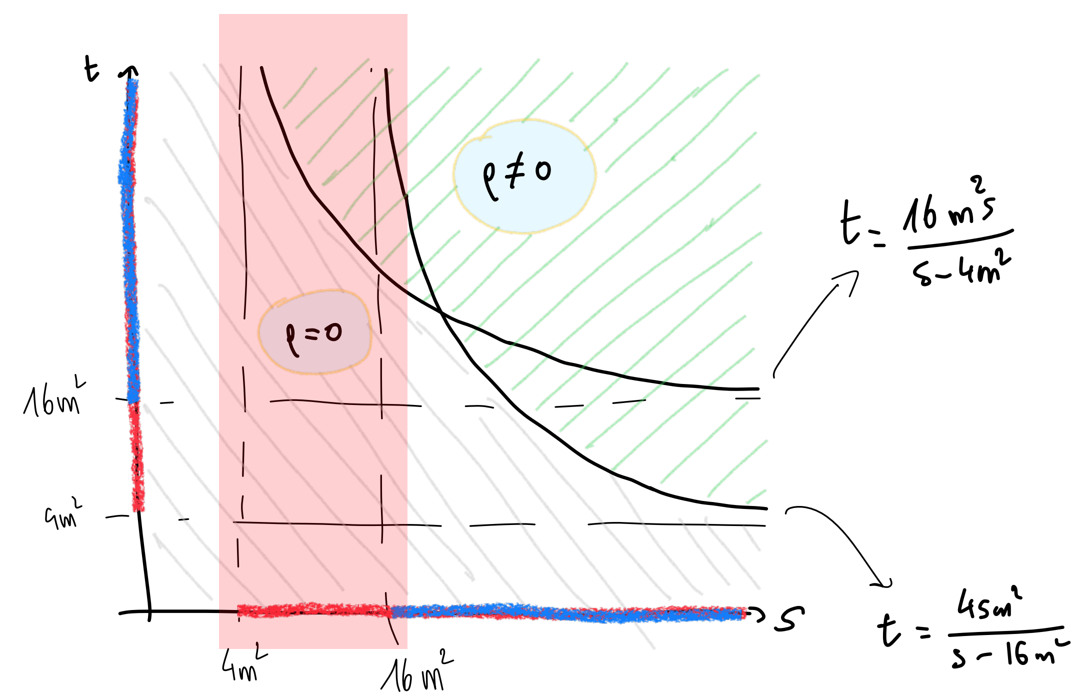
Support of double disc in (s,t)-plane

Elastic unitarity in 4d

Correira, Sever, Zhiboedov ‘20



elastic region



Support of double disc in (s,t)-plane

Elastic unitarity in 4d

Correira, Sever, Zhiboedov ‘20

- Consequences of elastic unitarity + crossing are profound
  - Aks’ theorem: “scattering implies production in d>2”.
  - Gribov’s theorem (disprove black disk diffraction model) $A_s(s,t) \neq s f(t)$ for $s \to \infty$
- As it seems, only one scheme was proposed in the literature to construct amplitudes which satisfy elastic unitarity + crossing, by Atkinson; [1968-1970].

Nucl.Phys. **B15** (1970) 331-331
A Proof of the Existence of Functions That Satisfy Exactly Both Crossing and Unitarity
D. Arkinson ⚠️

Nucl.Phys. **B15** (1970) 331-331
A Proof of the Existence of Functions That Satisfy Exactly Both Crossing and Unitarity (Ii) Charged Pions. No Subtractions
D. Atkinson

Nucl.Phys. **B13** (1969) 415-436
A Proof of the Existence of Functions That Satisfy Exactly Both Crossing and Unitarity (Iii). Subtractions
D. Atkinson

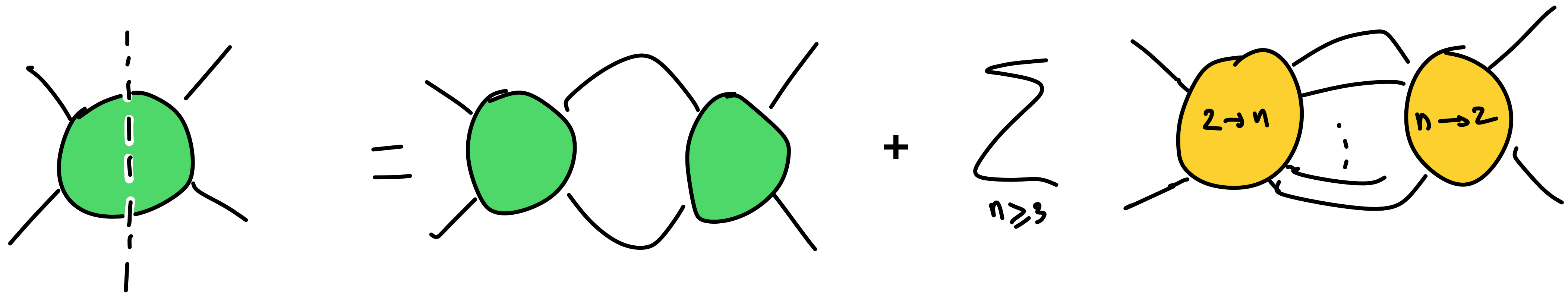
Nucl.Phys. **B23** (1970) 397-412
A Proof of the Existence of Functions That Satisfy Exactly Both Crossing and Unitarity. Iv. Nearly Constant Asymptotic Cross-Sections
D. Atkinson

Lecture notes:

S Matrix Construction Project: Existence Theorems, Rigorous Bounds and Models
D. Atkinson

Atkinson program

Recast *unitarity relations* as:



Atkinson program

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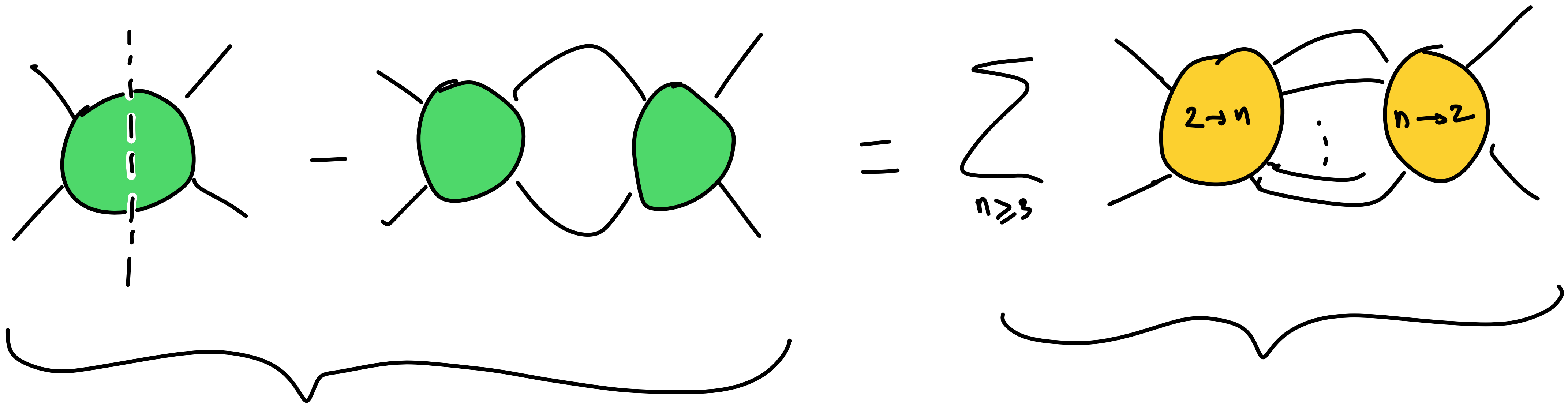

Scattering

output

Production

input

Atkinson program

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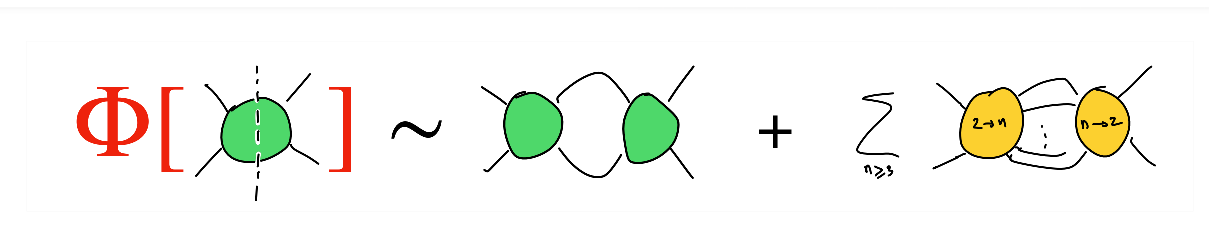
- Mathematical proofs of <u>existence</u> of functions that satisfy crossing, unitarity, elastic unitarity and Mandelstam analyticity, in d=4

- Let $\rho \sim \text{disc}_t \text{disc}_s T(s,t)$

- Proceeds by seeing unitarity equations as the fix point solutions of a map $\rho_* = \Phi[\rho_*]$ where $\Phi[\rho] \sim \int |\rho|^2 + v_{inel}$.



- He applied fix-point theorems (Leray-Schauder principle + contraction mapping principle), to show that the sequence $\rho_{n+1} = \Phi[\rho_n]$ converges to a unique solution for some range of ρ_0 and v_{inel}.

Atkinson program

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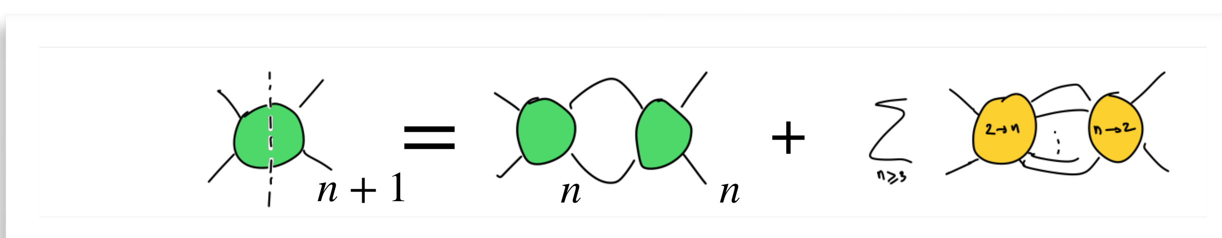

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Atkinson's proof

- Start from the map $\Phi : L \mapsto L$ where L is a Banach space of Hölder continuous functions

- Hölder continuity :
  $\forall x, y \in [0;1], |f(x) - f(y)| \leq k|x-y|^\alpha$
  for $0 < \alpha < 1$ and $k > 0$

- Define open ball $B = \{f \in L, \|f\| \leq b\}$ for some $b > 0$

- If $\Phi[B] \subset B$, Leray-Schauder principle
  $\implies \exists$ fixed point of Φ

- If Φ is *contracting*, i.e.
  $\|\Phi[f_1 - f_2]\| \leq c\|f_1 - f_2\|$, then the solution is also unique in B.

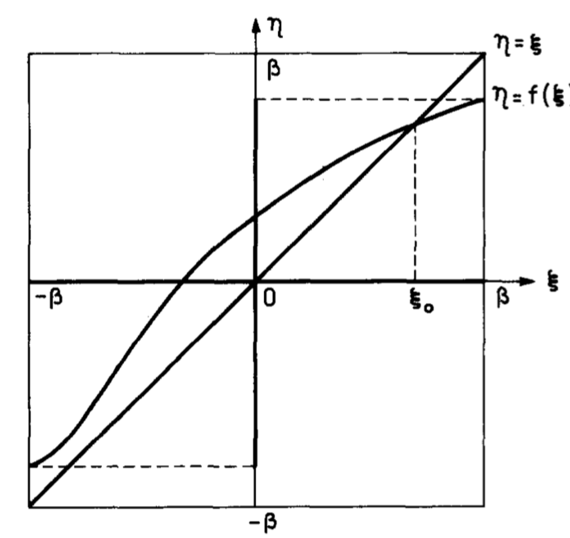

Fig. 1. Illustration of a fixed-point theorem. The image of the interval, $-\beta \leqslant \xi \leqslant \beta$, under the continuous, nonlinear mapping, f, is a subset of the same interval. Therefore the curve $\eta = f(\xi)$ intersects the line $\eta = \xi$ at least once, at a point ξ_0, such that $\xi_0 = f(\xi_0)$.

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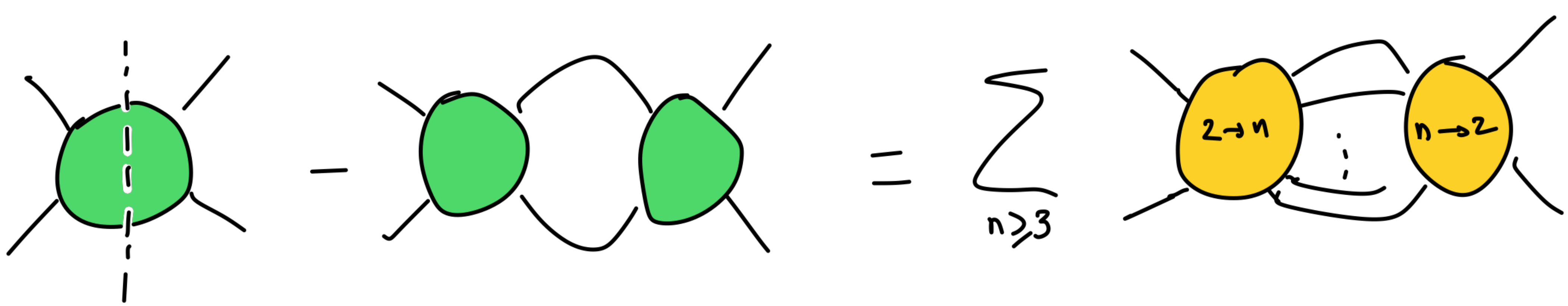
A Proof of the Existence of Functions That Satisfy Exactly Both Crossing and Unitarity
D. Arkinson

Inelastic function

- In practice we don't "choose" all of the $T_{2\to n}$.
  We choose a single function $v_{inel}(s,t) \sim \sum_{n\geq 3} |T_{2\to n}|^2$

- The problem is complete: allowing any functions allows to describe any amplitude

- Hence, there is a sense in which this philosophy is actually geared towards bootstrap

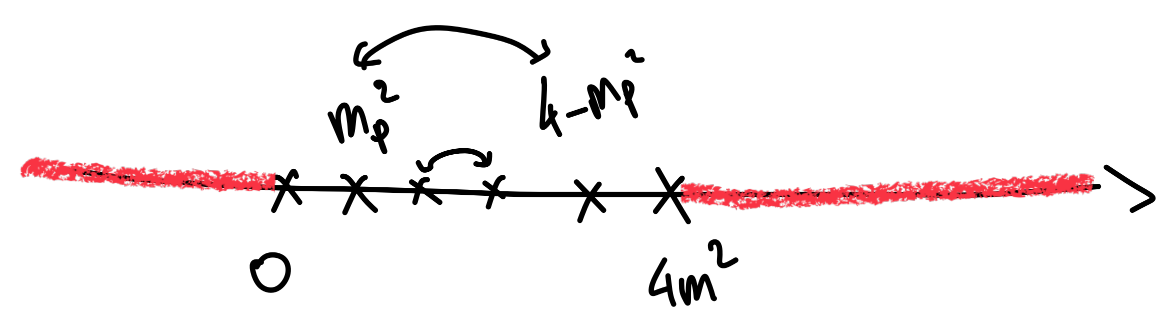


Atkinson’s program in 2d (=1+1)

S-matrices in d=2

Just one kinematic invariant: s, $(t = 0)$, $u = 4m^2 - s$.

Analyticity properties



Elastic unitarity $|S(s)| = 1, \quad 4m^2 \leq s < s_0$

Inelastic unitarity $|S(s)| \leq 1, \quad s \geq s_0$

$S(s)S^*(s) = 1 - f_{inel}(s)$

where $f_{inel}(s) = 0, s < s_0$

Crossing $S(s) = S(4m^2 - s)$

S-matrices in d=2

Elastic unitarity $|S(s)| = 1, \quad 4m^2 \leq s < s_0$

Inelastic unitarity $|S(s)| \leq 1, \quad s \geq s_0$

$\Rightarrow$ $S(s)S^*(s) = 1 - f_{inel}(s)$

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Crossing $S(s) = S(4m^2 - s)$

In terms of T:

$$S(s) = 1 + i\frac{T(s)}{\sqrt{s(s-4m^2)}}$$

S-matrices in d=2

Elastic unitarity	$\lvert S(s)\rvert = 1, \quad 4m^2 \le s < s_0$	$S(s)S^*(s) = 1 - f_{inel}(s)$
Inelastic unitarity	$\lvert S(s)\rvert \le 1, \quad s \ge s_0$	where $f_{inel}(s) = 0, s < s_0$
Crossing	$S(s) = S(4m^2 - s)$	

In terms of T:

$$S(s) = 1 + i\frac{T(s)}{\sqrt{s(s-4m^2)}}$$

$$\hat{S} = \hat{1} \times S(s) = \hat{1} + i\,\delta^{(2)}(p_1 + p_2 - p_3 - p_4)\,T(s)$$

$$\hat{1} \sim 4E_1E_2\,\delta(\vec{p}_1 - \vec{p}_3)\,\delta(\vec{p}_2 - \vec{p}_4) + (3 \leftrightarrow 4) \quad \left(\begin{array}{l} 1 \longrightarrow 3 \\ 2 \longrightarrow 4 \end{array}\right) + \not{X}$$

$$\hat{1} + i\,\delta^{(2)}\,T(s) = \hat{1}\underbrace{\left(1 + i\frac{T(s)}{\sqrt{s(s-4m^2)}}\right)}_{= S(s)}$$

S-matrices in d=2

Elastic unitarity $|S(s)| = 1, \quad 4m^2 \leq s < s_0$

Inelastic unitarity $|S(s)| \leq 1, \quad s \geq s_0$

$\Rightarrow$ $S(s)S^*(s) = 1 - f_{inel}(s)$

where $f_{inel}(s) = 0, s < s_0$

Crossing $S(s) = S(4m^2 - s)$

In terms of T:

$$S(s) = 1 + i\frac{T(s)}{\sqrt{s(s-4m^2)}}$$

$$v_{inel}(s) = f_{inel}(s)\sqrt{s(s-4m^2)}/4$$

$$\mathfrak{I}T(s) = \frac{1}{2\sqrt{s(s-4m^2)}}|T(s)|^2 + v_{inel}(s)$$

Our problem:

Given v_{inel}, find T(s) that satisfies Mandelstam analyticity, crossing, elastic unitarity and inelastic unitarity.

Will solve by:

1. searching fixed point of map Φ defined by

   Fixed-point iteration

$$\Phi[\Im T(s)] = \frac{1}{2\sqrt{s(s-4m^2)}} |T(s)|^2 + v_i(s)$$

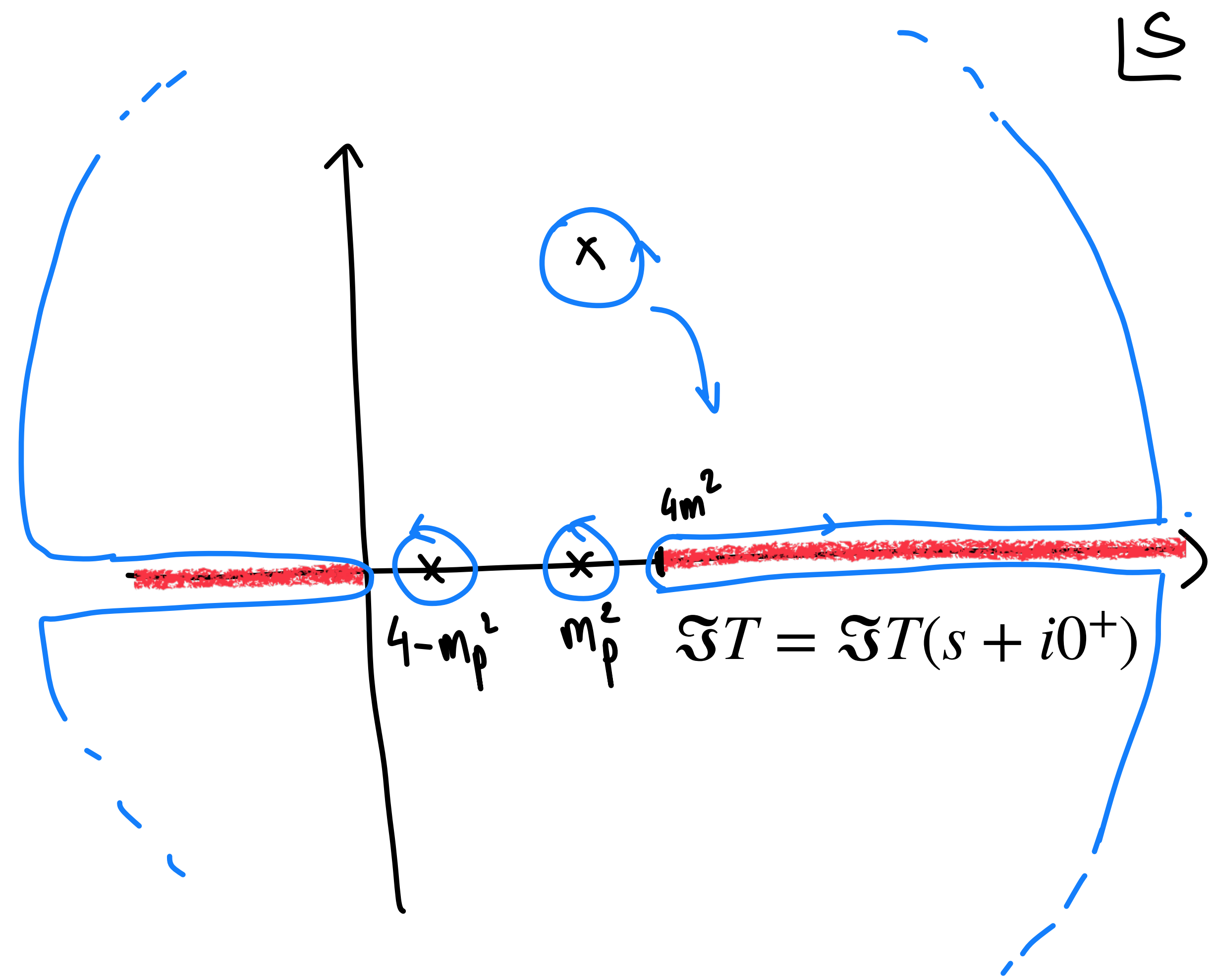
2. searching root of $\Psi[f] = f - \Phi[f]$

   Newton method

Note that Φ implicitly contains a step $\Im T \to \Re T$, given by a dispersion integral

Dispersion integral

$$T_n(s) = c_\infty - \frac{g^2}{s - m_p^2} - \frac{g^2}{4m^2 - s - m_p^2} + \int_{4m^2}^{\infty} \frac{ds'}{\pi} \Im T_n(s') \left(\frac{1}{s' - s} + \frac{1}{s' - (4m^2 - s)} \right)$$



$\Im T = \Im T(s + i0^+)$

Dispersion integral

$$\Re T_n(s) = c_\infty - \frac{g^2}{s - m_p^2} - \frac{g^2}{4m^2 - s - m_p^2} + P.V. \int_{4m^2}^{\infty} \frac{ds'}{\pi} \Im T_n(s') \left(\frac{1}{s' - s} + \frac{1}{s' - (4m^2 - s)} \right)$$

Problem: defined in this way, $\Re T_n(4m^2) \neq 0$

$$\implies \Im T_{n+1}(s) \underset{s \to 4}{\to} \infty$$

which leads to a divergent dispersion integral at next step

$$\Phi[\Im T(s)] = \frac{1}{2\sqrt{s(s - 4m^2)}} |T(s)|^2 + v_i(s)$$

Dispersion integral

- But we actually know the near-threshold behaviour of unitarity equations. Not hard to see that

$$\Im T(s) \sim_{s\to 4} (s-4m^2)^{k/2} \text{ with } k \geq 1$$

- So we can force that $\Re T_n(4)$ vanishes, by defining g such that

$$\Re T_n(4m^2) = 0$$
$$= c_\infty - \frac{g_n^2}{s-m_p^2} - \frac{g_n^2}{4m^2 - s - m_p^2} + P.V.\int_{4m^2}^{\infty} \frac{ds'}{\pi} \Im T_n(s') \left(\frac{1}{s'-4m^2} + \frac{1}{s'} \right)$$

Our map

Iterative solution:

$$\operatorname{Im} T_{n+1}(s) = \begin{cases} \Phi(\operatorname{Im} T_n) & \text{(fixed-point iteration)} \quad (2.22a) \\ \operatorname{Im} T_n - (\Psi')^{-1} \cdot \Psi(T_n) & \text{(Newton-Kantorovich method)} \quad (2.22b) \end{cases}$$

$$T_{n+1}(s) = c_\infty - \frac{g_{n+1}^2}{s - m_p^2} - \frac{g_{n+1}^2}{4m^2 - s - m_p^2} + \int_{4m^2}^{\infty} \frac{ds'}{\pi} \operatorname{Im} T_{n+1}(s') \left(\frac{1}{s' - s} + \frac{1}{s' - (4m^2 - s)} \right) \tag{2.23}$$

$$g_{n+1}^2 = \left(\frac{1}{4m^2 - m_p^2} - \frac{1}{m_p^2} \right)^{-1} \left(c_\infty + \int_{4m^2}^{\infty} \frac{ds'}{\pi} \operatorname{Im} T_{n+1}(s') \left(\frac{1}{s' - 4m^2} + \frac{1}{s'} \right) \right) \tag{2.24}$$

Input data:

- mass of the bound state m_p
- inelasticity v_{inel}
- constant at infinity c_∞

Iterates:

- imaginary part of the amplitude on the cut

Analytic solution

Analytic solution

is known already

so, just to make sure that you don't waste brain
computing time being confused by this:

this is a new numerics method to solve a solved problem [arXiv:1607.06110] JHEP **1711** (2017) 143

The S-matrix Bootstrap II: Two Dimensional Amplitudes = *PPTvRV '16*

M. F. Paulos, J. Penedones, J. Toledo, B. C. van Rees, P. Vieira

PPTvRV '16

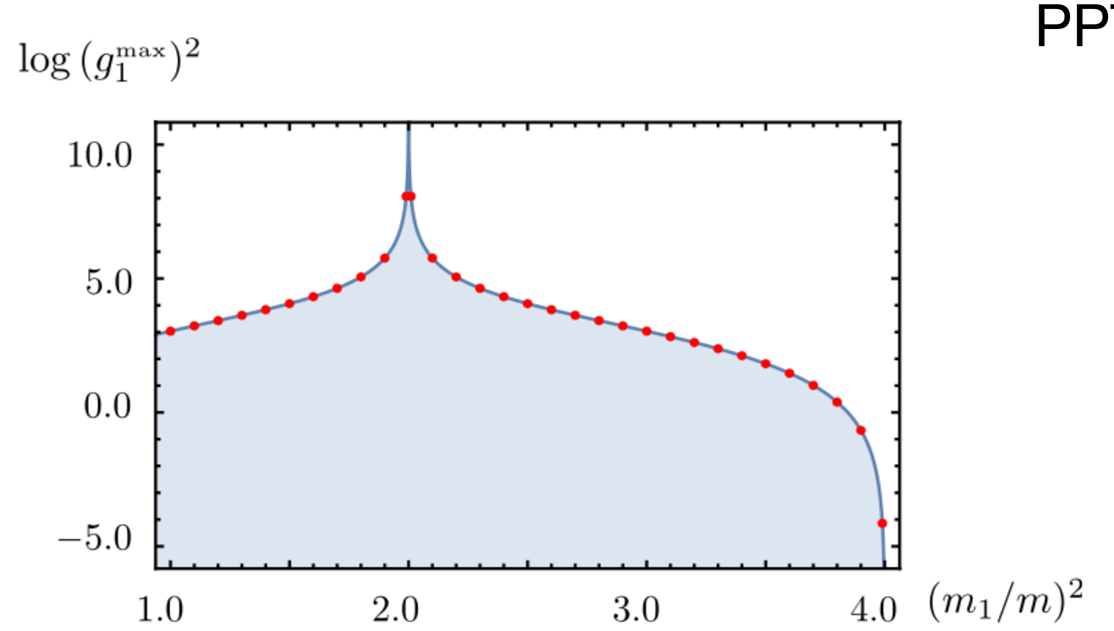


Figure 4: Maximum cubic coupling $g_1^{\max}$ between the two external particles of mass m and the exchanged particle of mass m_1. Here we consider the simplest possible spectrum where a single particle of mass m_1 shows up in the elastic S-matrix element describing the scattering process of two mass m particles. The red dots are the numerical results. The solid line is an analytic curved guessed above (9) and derived in the next section. The blue (white) region corresponds to allowed (excluded) QFT's for this simple spectrum.

Analytic solution

PPTvRV '16

- It turns out that in 2d, an exact solution can be written

$$S(s) = S_{\text{elastic}}(s)\, e^{\int_{4m^2}^{\infty} \frac{ds'}{2\pi i} \log(1-f_i(s'))\sqrt{\frac{s(s-4m^2)}{s'(s'-4m^2)}}\left(\frac{1}{s'-s}+\frac{1}{s'-(4m^2-s)}\right)}$$

- S_{elastic} is only defined by demanding $|S_{\text{elastic}}| = 1$
- This introduces an ambiguity that played an important role in our analysis: *given inelasticity, there is an infinite freedom to choose* S_{elastic}
- *<u>Remark:</u>* a priori absent in 4d because no such purely elastic amplitudes should exist (Aks' theorem)

Elastic S-matrices

- No particle production ⟶ *integrable* theories (See review by P Dorey) [hep-th/9810026]

- Spanned by *CDD* S-matrices (Castillejo-Dalitz-Dyson)

$$S_{\mathrm{CDD}}(s, m_0) = \frac{\sqrt{s(4m^2 - s)} \pm \sqrt{m_0^2(4m^2 - m_0^2)}}{\sqrt{s(4m^2 - s)} \mp \sqrt{m_0^2(4m^2 - m_0^2)}}$$

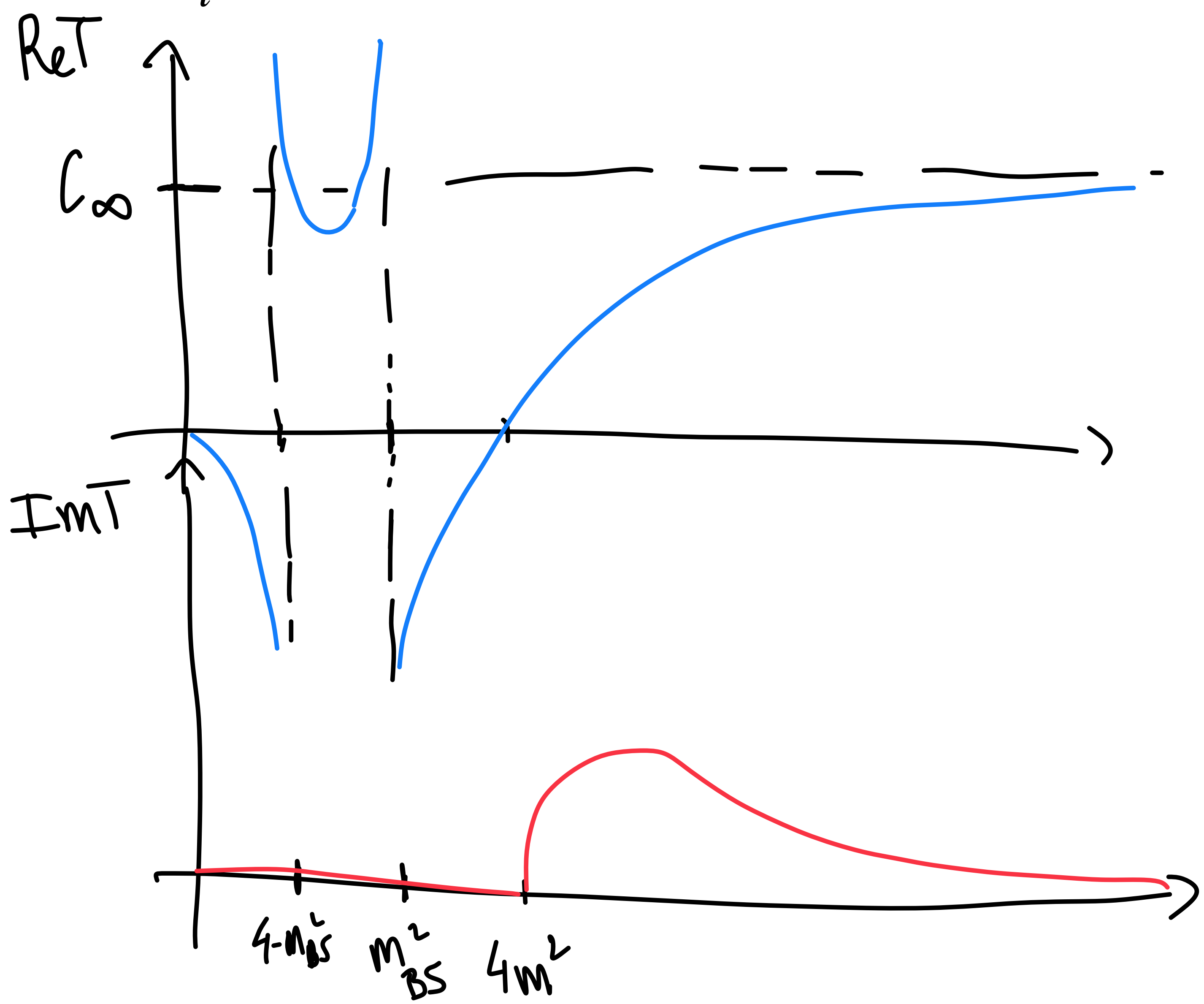
+ : pole
- : zero

$$S_{elastic}(s) = \prod_i S_{CDD}(s, m_i)$$

Elastic S-matrices

$$S_{elastic}(s) = \prod_i S_{CDD}(s, m_i)$$

$$S_{\text{elastic}}(s) = 1 + i\frac{T_{\text{elastic}}(s)}{\sqrt{s(s-4m^2)}}$$



Elastic S-matrices

$$S_{elastic}(s) = \prod_i S_{CDD}(s, m_i) \qquad S_{\text{elastic}}(s) = 1 + i\frac{T_{\text{elastic}}(s)}{\sqrt{s(s-4m^2)}}$$

- The corresponding amplitudes $T_{\text{elastic}}(s)$ go to constants at infinity given by

$$\lim_{s\to\infty} T_{\text{elastic}}(s) = c_\infty = 2\sum_{j=1}^{N_{poles}} \sqrt{m_{p_j}^2(4m^2 - m_{p_j}^2)} - 2\sum_{j=1}^{N_{zeros}} \sqrt{m_{z_j}^2(4m^2 - m_{z_j}^2)}$$

- At fixed pole locations, many amplitudes can still have the same c_∞, by adjusting the **number** or **position** of the zeros.

- remark: zeros decreases the constant at infinity

Results

Numerical strategies

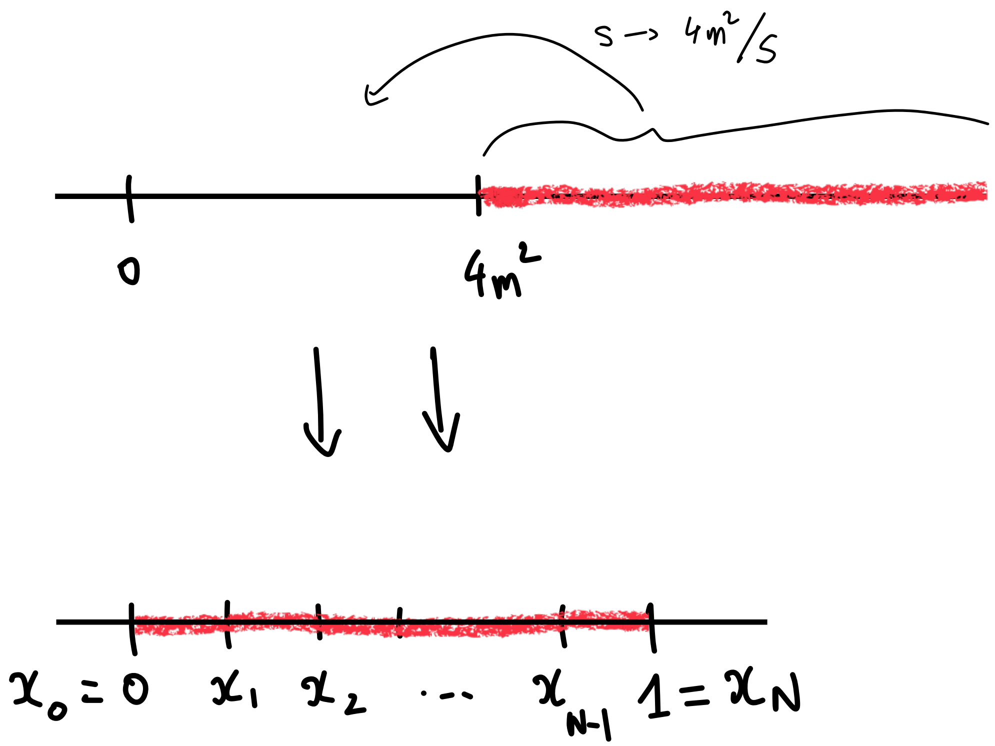
1. Fixed-point iteration

2. Newton's method

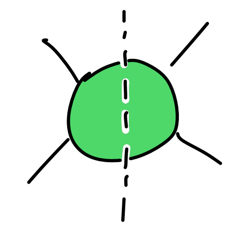
remark: everything was done with Mathematica

Discretization

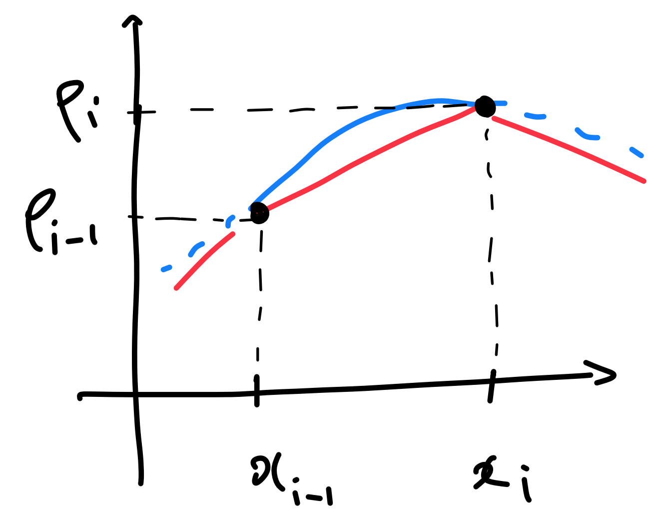
- Variable $x = 4/s \in [0,1]$, grid $x_0 = 0, \ldots, x_i, x_N = 1$



Interpolation

$$\rho(s) := \Im T(s) =$$ 

- Linear interpolant
- Bernstein polynomials interpolant

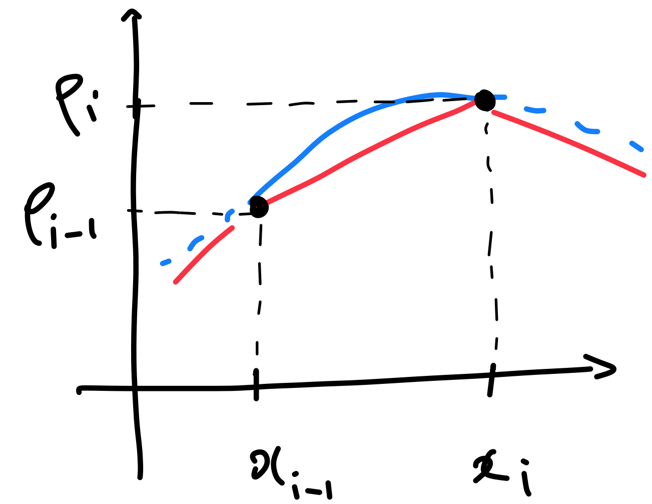


PPTvRV '16

$$\rho(x) = \rho_{i-1} + (\rho_i - \rho_{i-1})\frac{x - x_{i-1}}{x_i - x_{i-1}}, \quad x_{i-1} < x < x_i$$

Interpolation

- Linear interpolant
- Bernstein polynomials interpolant



PPTvRV '16

$\rightarrow$ discrete version of the dispersion integral:

$$\rho(x) = \rho_{i-1} + (\rho_i - \rho_{i-1})\frac{x - x_{i-1}}{x_i - x_{i-1}}, \quad x_{i-1} < x < x_i$$

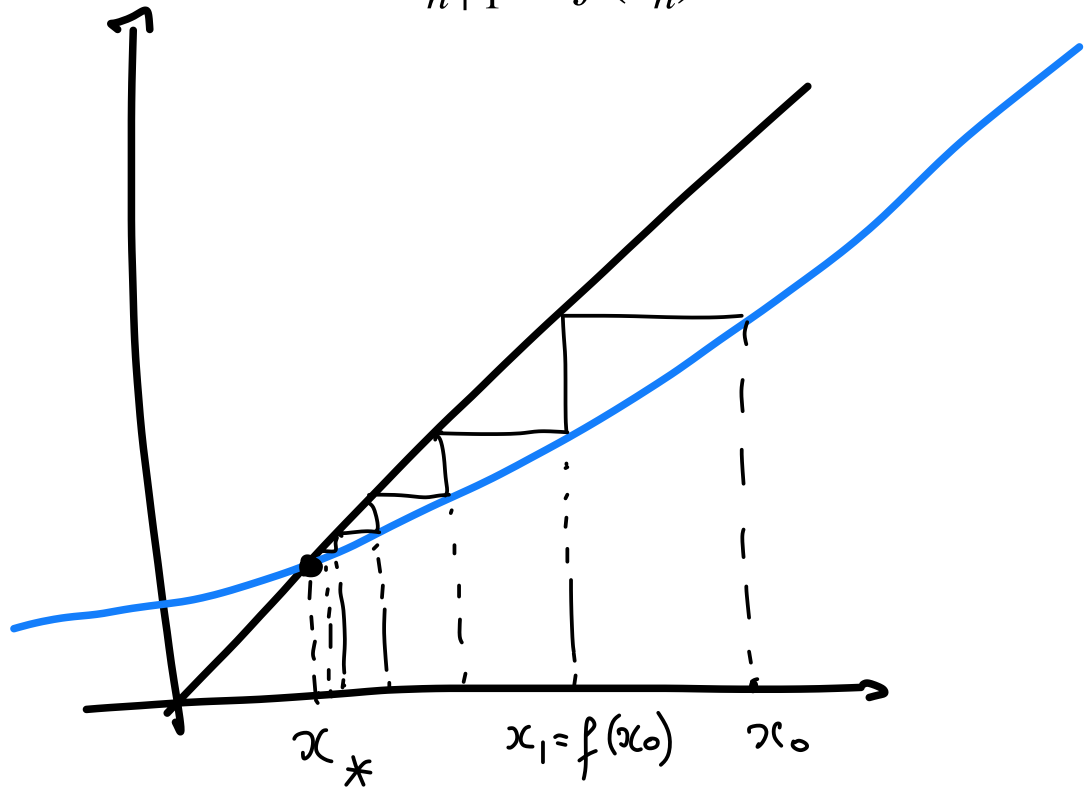
$$\int_4^\infty \rho(s')\left(\frac{1}{s' - 4/x_i} + \frac{1}{s' - (4 - 4/x_i)}\right) ds' \rightarrow \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \rho(s')\left(\frac{1}{s' - 4/x_i} + \frac{1}{s' - (4 - 4/x_i)}\right) ds'$$

$$= \sum_{j=1}^{N} B_{i,j}\rho_j$$

Fixed-point iteration

Fixed-point iteration

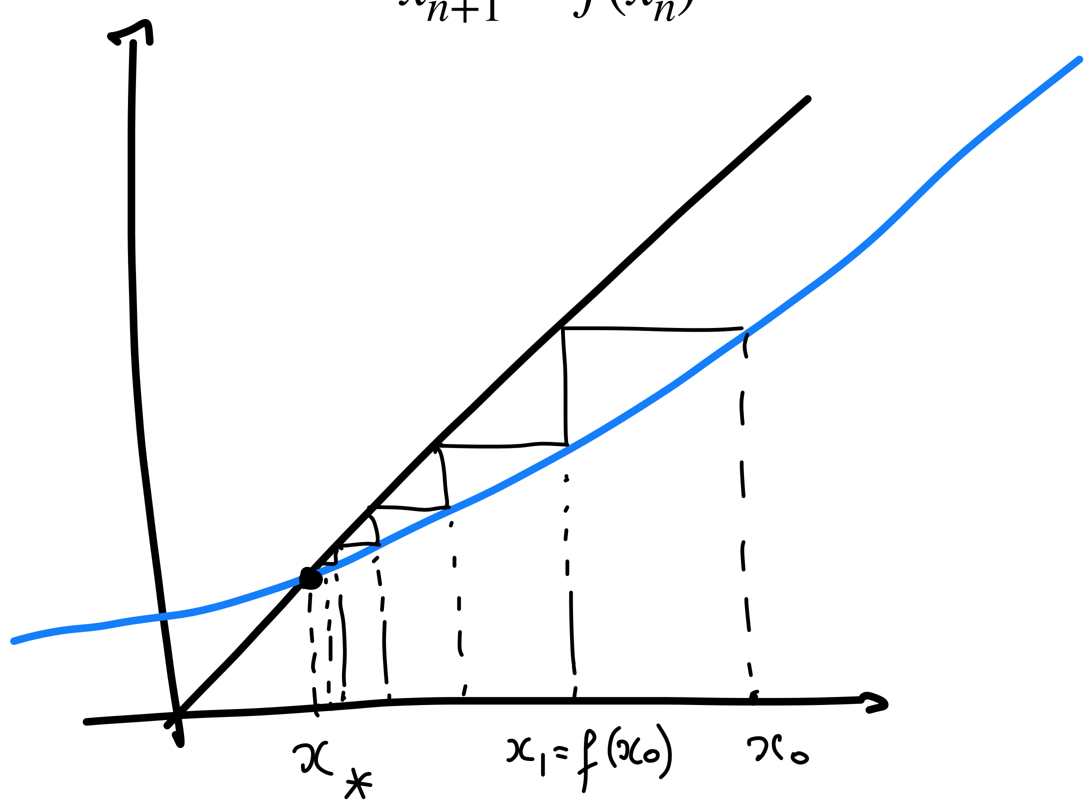
$$x_{n+1} = f(x_n)$$



Fixed-point iteration

The Babylonian method

$$x_{n+1} = f(x_n)$$



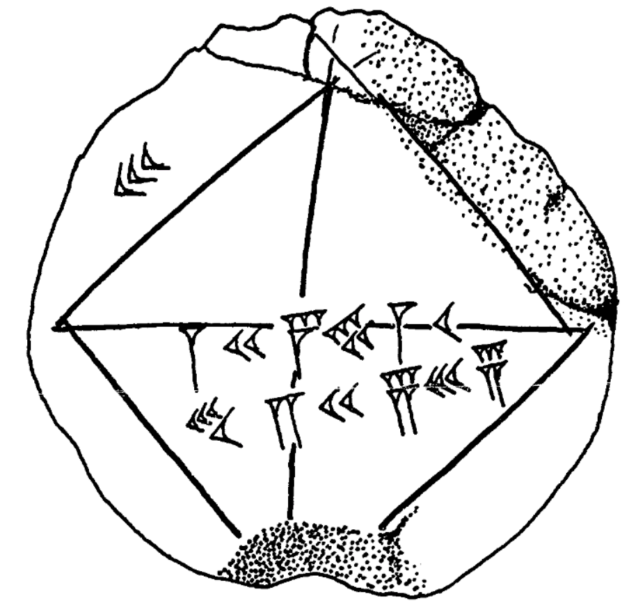


FIG. 1. The Old Babylonian tablet YBC 7289. (From Asger Aaboe, *Episodes from the Early History of Mathematics*, Washington, DC: The Mathematical Association of America, 1964. Reprinted by permission of the Mathematical Association of America.)

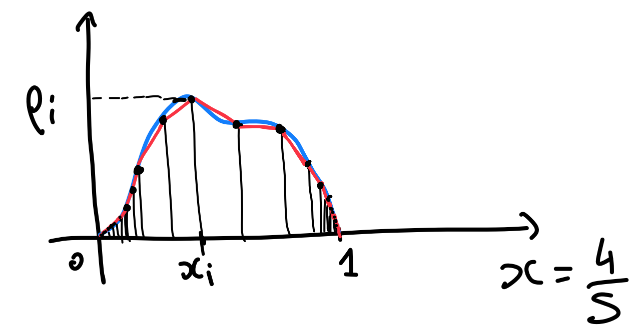
Fowler, Robson 1998

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

Babylonian scribes computing $\sqrt{2}$ (perhaps)

Fixed-point iteration

1. Discretization
2. Interpolation
3. Dispersion integral
4. Iteration



Im T_n^2 Re T_n^2

$$\rho_{n+1,i} = \Phi(\rho_{n,j})_i := \frac{x_i}{8\sqrt{1-x_i}}\left(\rho_{n,i}^2 + (G_{ij}\cdot\rho_{n,j} + q_i)^2\right) + v_{inel}(x_i)$$

$$\Re T_{n,i} = c_\infty - g_n^2\left(\frac{1}{4/x_i - m_p^2} - \frac{1}{4/x_i - (4-m_p^2)}\right) + \frac{1}{\pi}\sum_{j=1}^{N-1} B_{ij}\rho_{n,j}$$

$$g_n^2 = \left(\frac{1}{4-m_p^2} - \frac{1}{m_p^2}\right)^{-1}\left(\frac{1}{\pi}\sum_j B_{Nj}\rho_{n,j} + c_\infty\right)$$

$$G_{n,ij} = B_{ij} - \frac{P(x_i)}{P(1)}B_{Nj}, \quad q_i = c_\infty\left(1 - \frac{P(x_i)}{P(1)}\right),$$

$$P(x) \equiv \frac{1}{4/x - m_p^2} - \frac{1}{4/x - (4-m_p^2)}$$

Fixed-point iteration

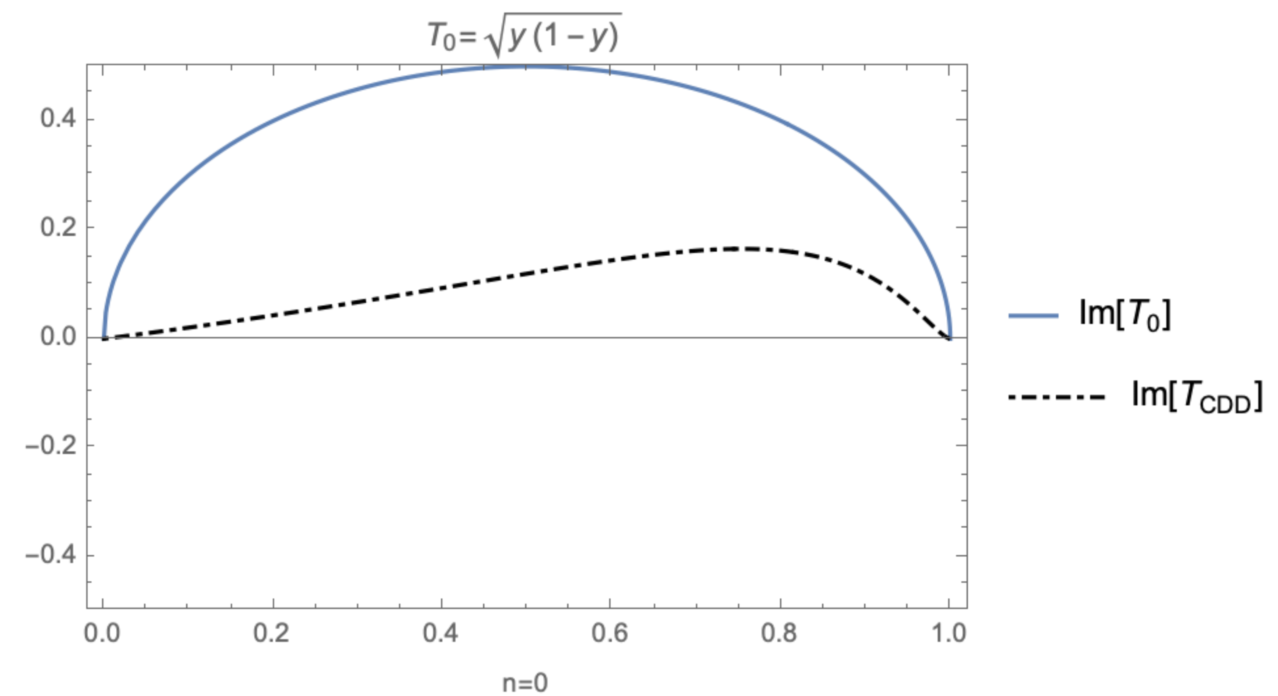
$$\rho_{n+1,i} = \Phi(\rho_{n,j})_i := \frac{x_i}{8\sqrt{1-x_i}}\left(\rho_{n,i}^2 + (G_{ij}\cdot\rho_{n,j} + q_i)^2\right) + v_{inel}(x_i) \qquad \textbf{(1)}$$

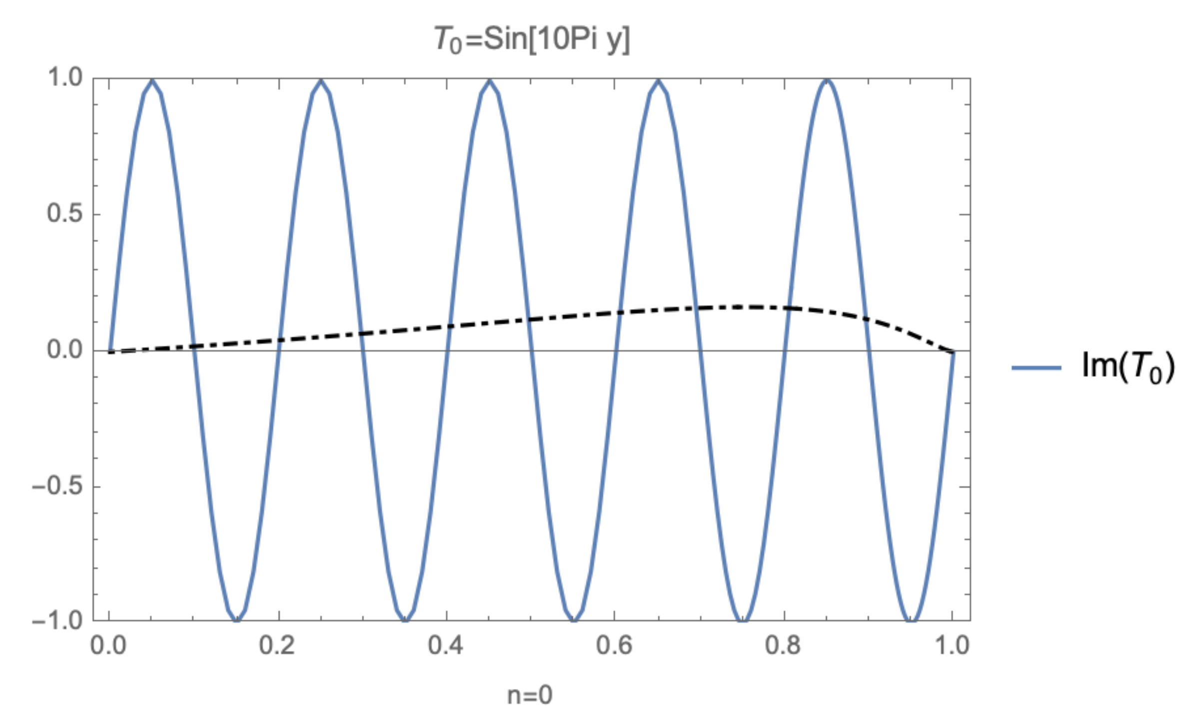
1. Remark: In general, we want to find solutions of **(1)** without the n index.
2. **Whichever way that works is good.**
3. In the context of **(1)**, what takes longest is to pre-compute the matrix $G_{i,j}$ (order of minutes to hours depending on grid size N)
4. The map, defined as it is, encodes everything : unitarity (elastic & inelastic), analyticity, and crossing.

Now, back to fixed-point iteration $\rho_{n+1} = \Phi[\rho_n]$

Results: one-pole amplitudes

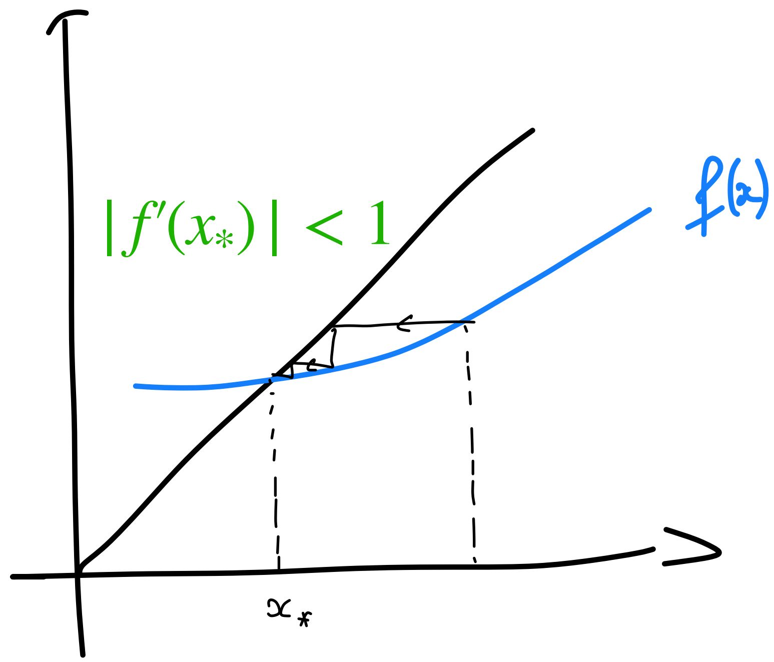
- Inputs: c_∞, v_{inel}, m_p
- Converge to 1-pole 1-zero amplitudes
- Independently of starting point (granted not too big)
- Ceases to converge when either inelasticity, or c_∞ becomes too big.



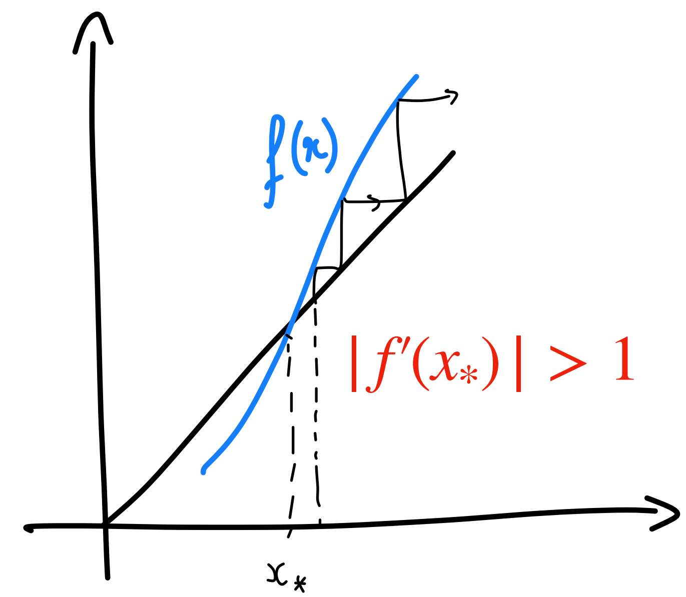


Convergence of fixed-point: spectral radius

<u>In 1d:</u>



converges



diverges

<u>In N-d:</u>

- <u>Def:</u> the *spectral radius* of a bounded linear operator is its maximal eigenvalue, in modulus.

- For a map $\Phi : \mathbb{R}^N \mapsto \mathbb{R}^N$, in a neighbourhood of a solution $\rho_* = \Phi[\rho_*]$, you converge to a unique solution whenever the *spectral radius* of the Jacobian of the map $J_{ij} = \partial_i \Phi[\rho_*]/\partial \rho_j$ is smaller than one, |J|<1

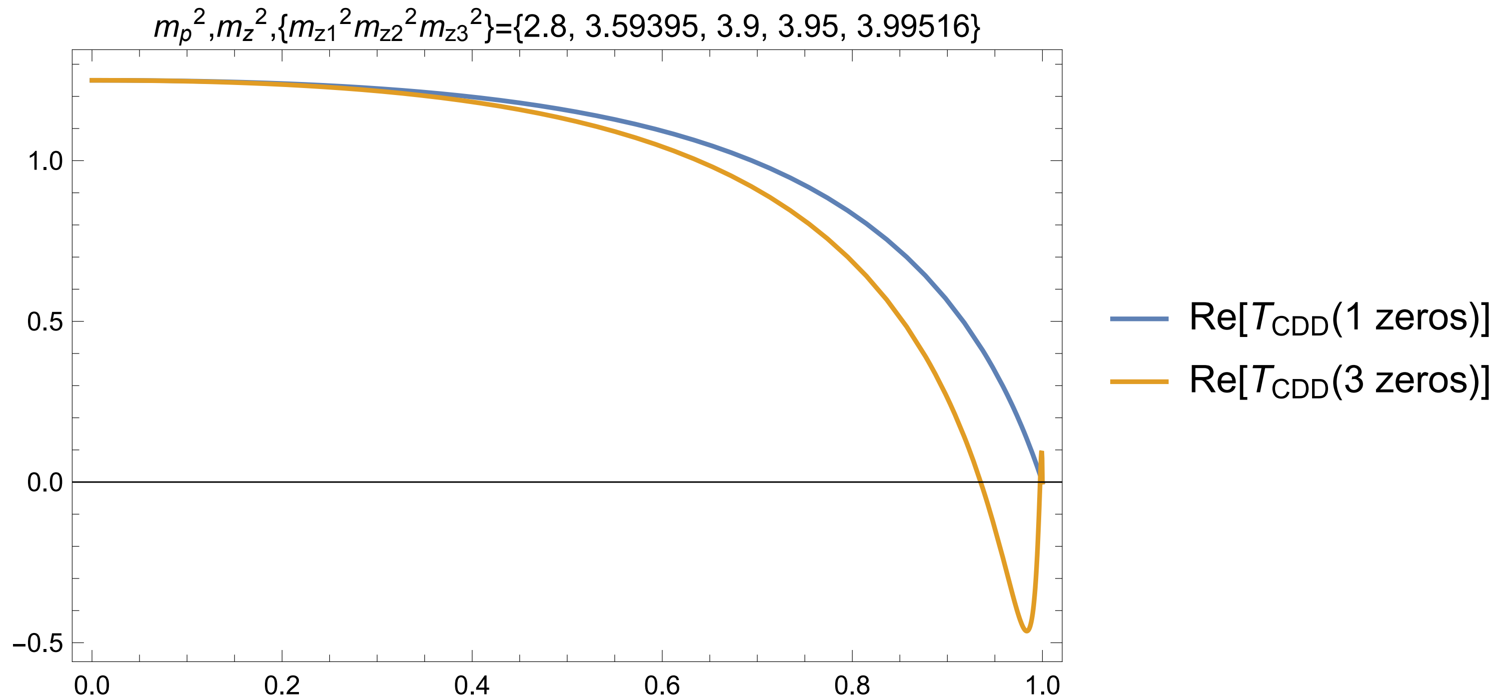
Divergence on 1-pole 3-zero

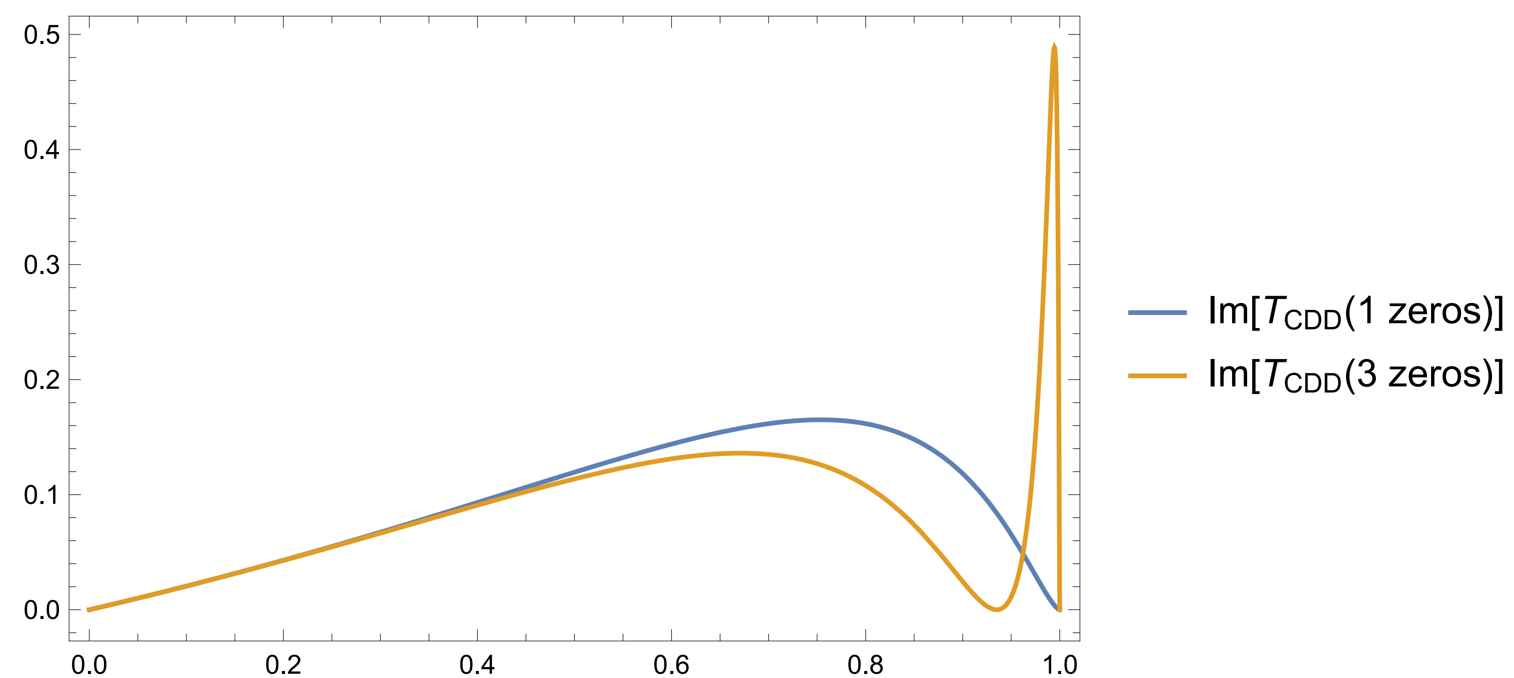
one-zero and three-zero

have the same inputs as

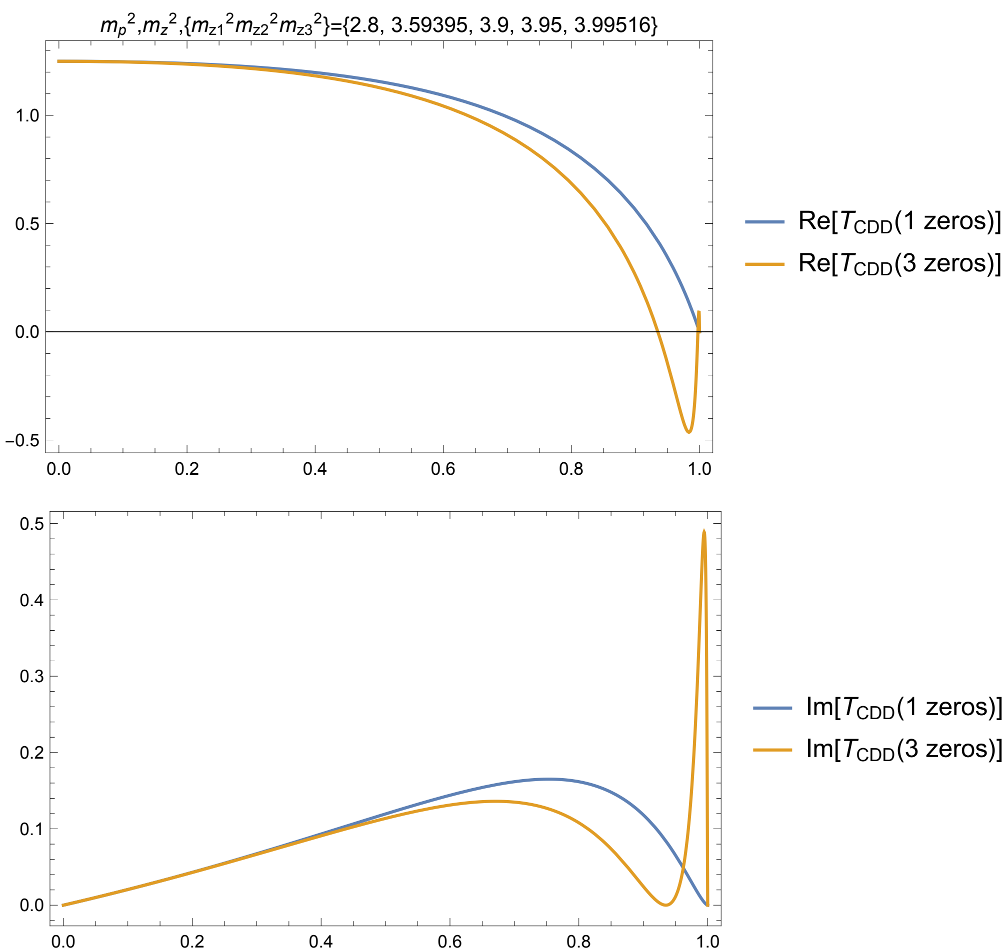
far as the algorithm is concerned:

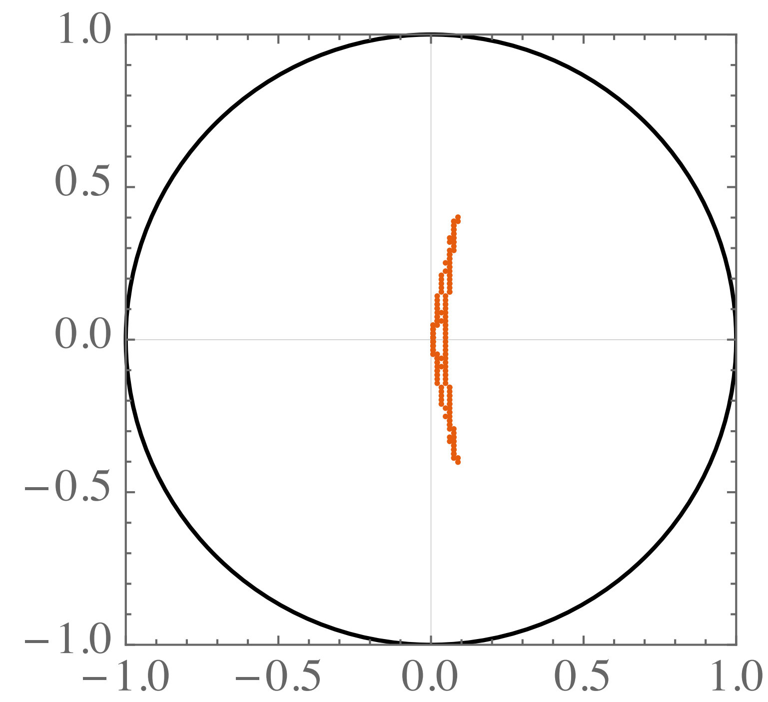
same c_∞, same pole and $v_{inel} = 0$



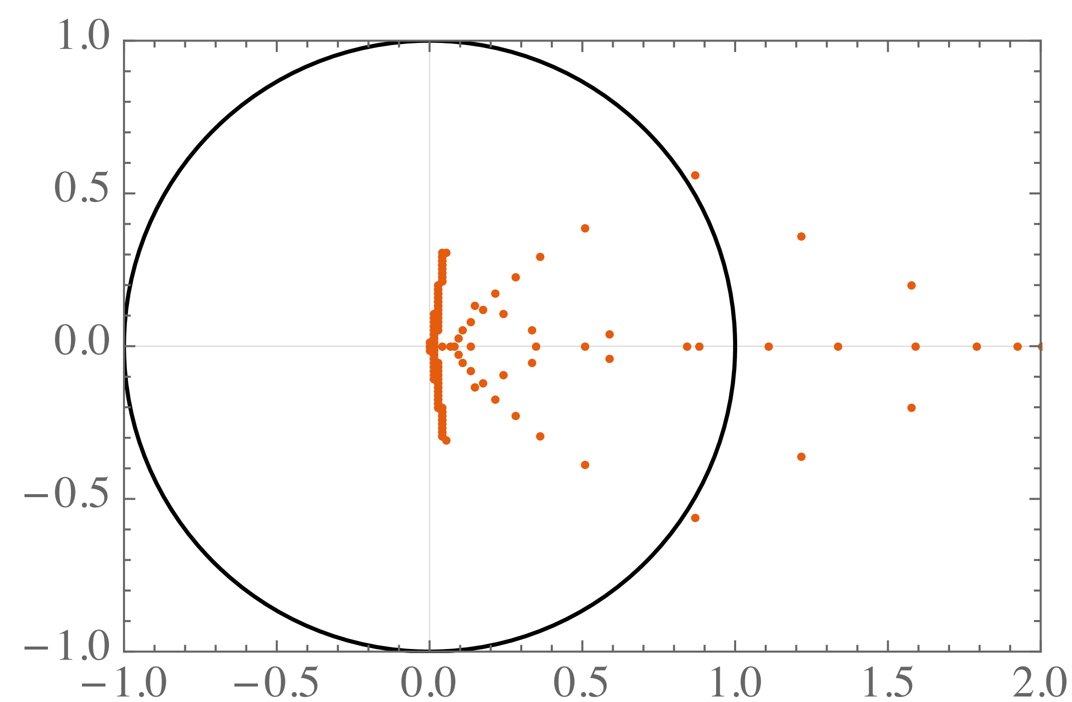


Divergence on 1-pole 3-zero



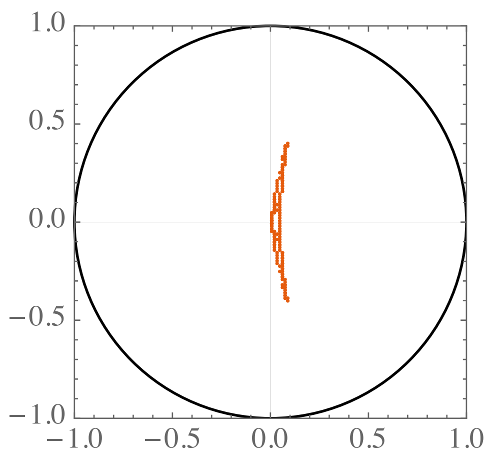


Jacobian's e.v. for 1-pole 1-zero

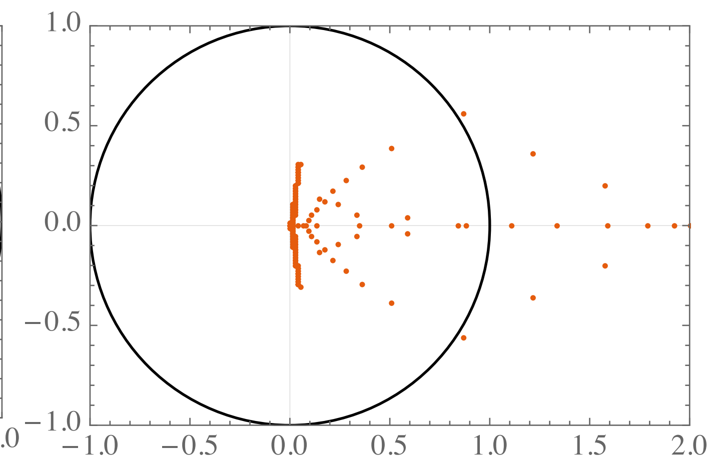


Jacobian's e.v. for 1-pole 3-zero

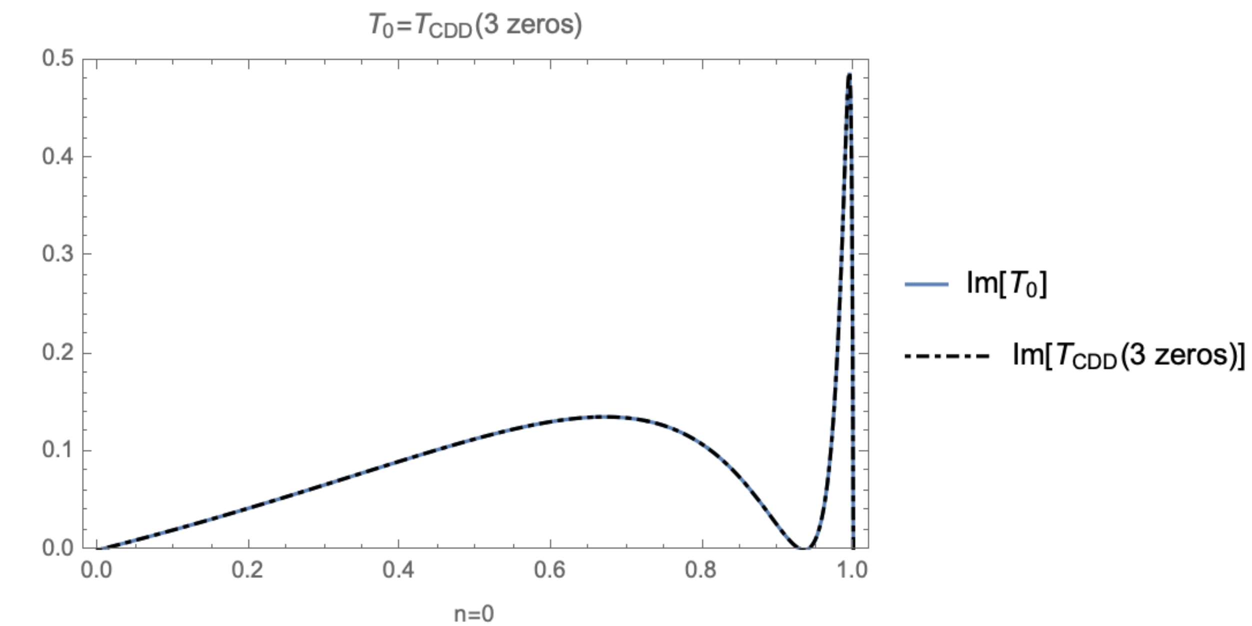
Divergence on 1-pole 3-zero

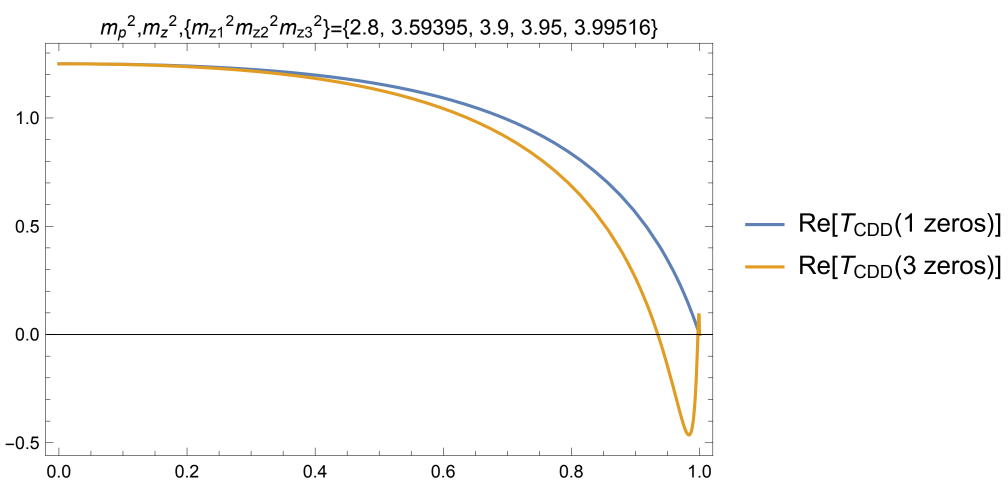


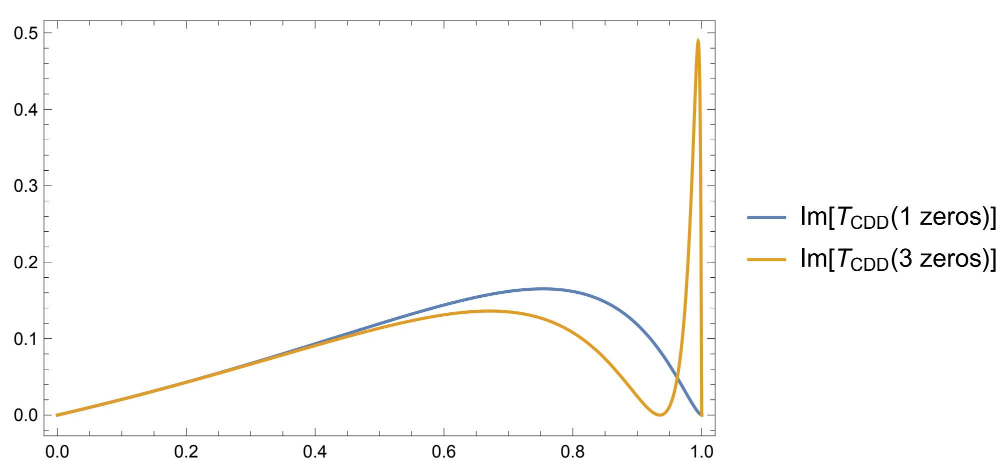
Jacobian's e.v. for 1-pole 1-zero

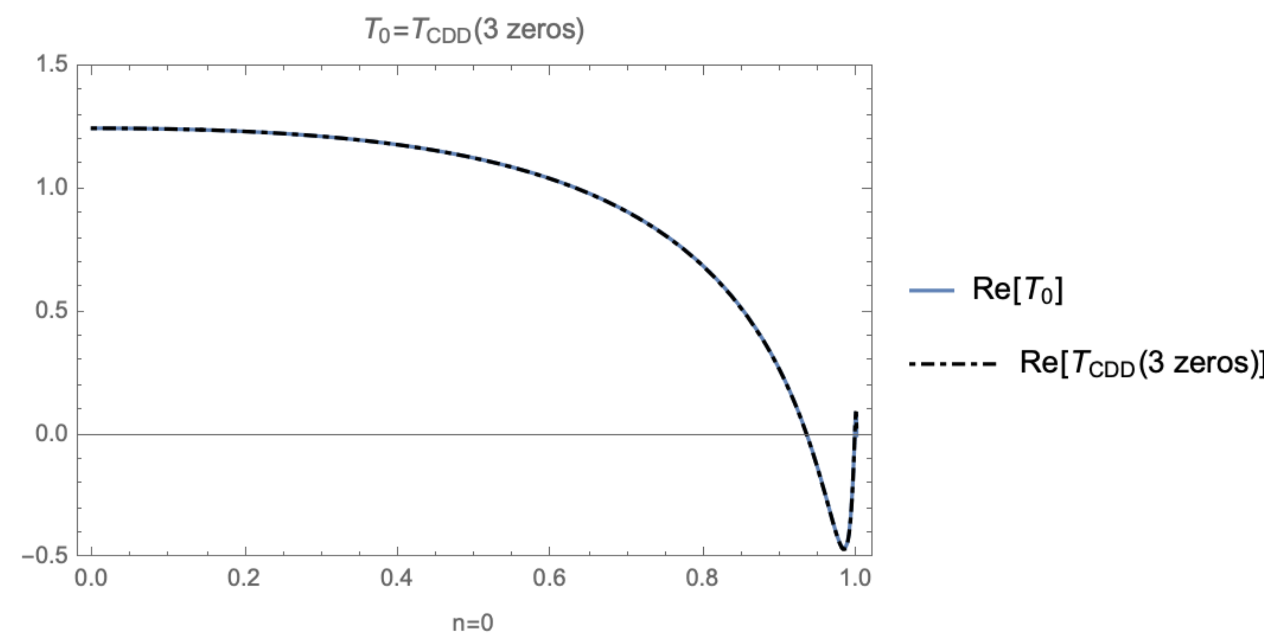


Jacobian's e.v. for 1-pole 3-zero



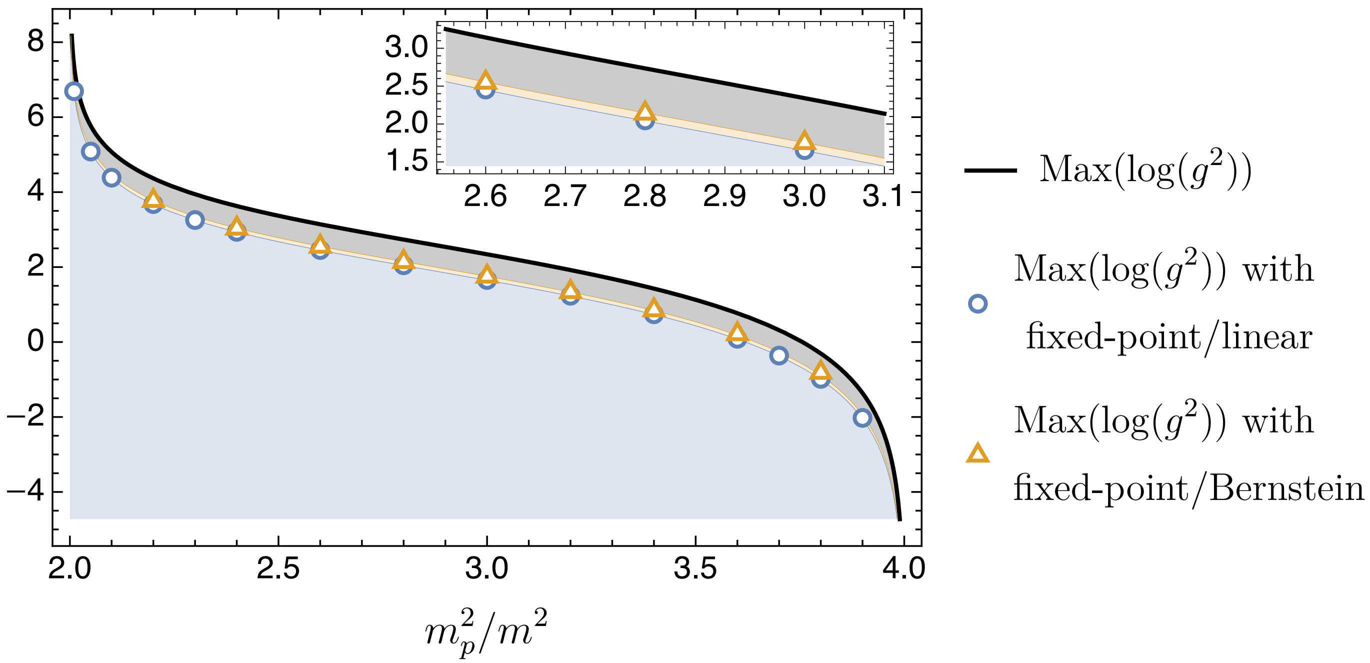


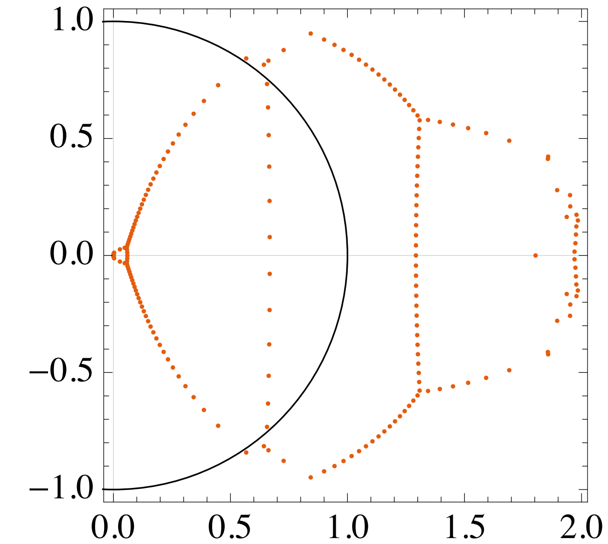




Summary of fixed-point results

- Converges on *n*-pole *n*-zero amplitudes
- Diverges on *n*-pole *m*-zero amplitudes with $n \neq m$
- On 1-pole 1-zero amplitudes, we can fill almost all of function space, as represented by the coupling, except for a small band





e.v.'s at the maximal coupling clearly above 1

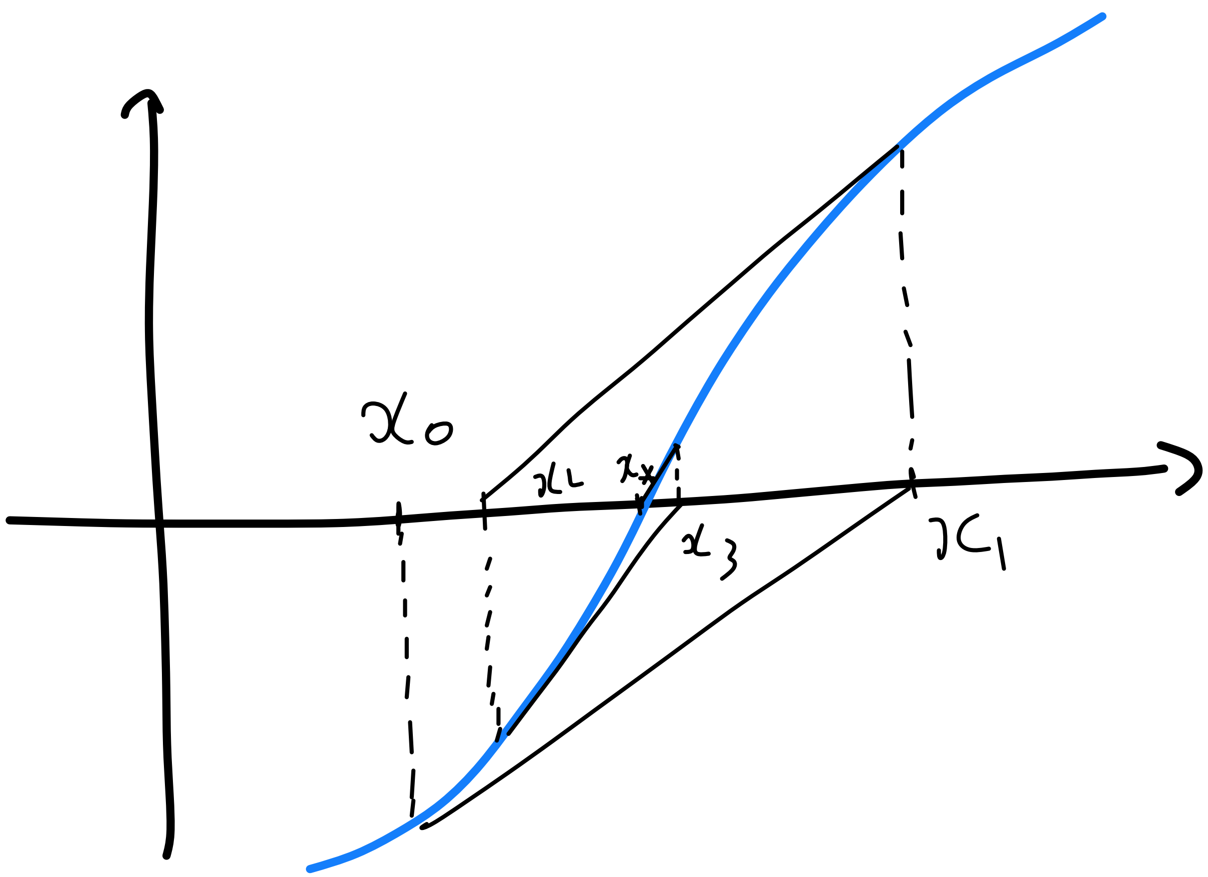
Newton’s method

Newton's method

Newton-Raphson

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

$$g(x) = x - f(x)$$



"Oh, Diamond! Diamond! thou little knowest what mischief thou hast done!"

Newton's method

1. Discretization on the grid
2. Interpolation
3. Dispersion integral
4. Iteration $\rho_{n+1,i} = \rho_{n,i} - (J^{\Psi})^{-1} \cdot (\rho_n - \Phi(\rho_n))_i$ matrix inversion: slow ⚠️

$$J^{\Psi} \cdot (\rho_{n+1,i} - \rho_{n,i}) = \rho_{n,i} - \Phi(\rho_n)_i$$

way faster ✅

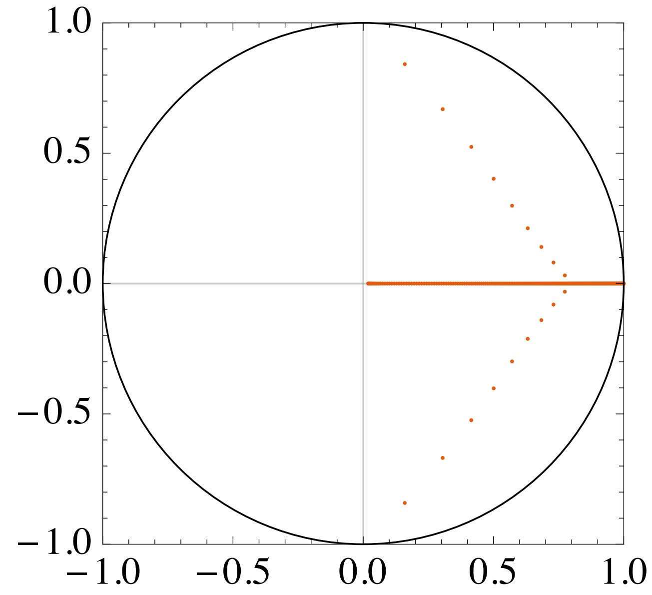
LinearSolve[m, b]
finds an x that solves the matrix equation $m.x == b$.

Newton's method

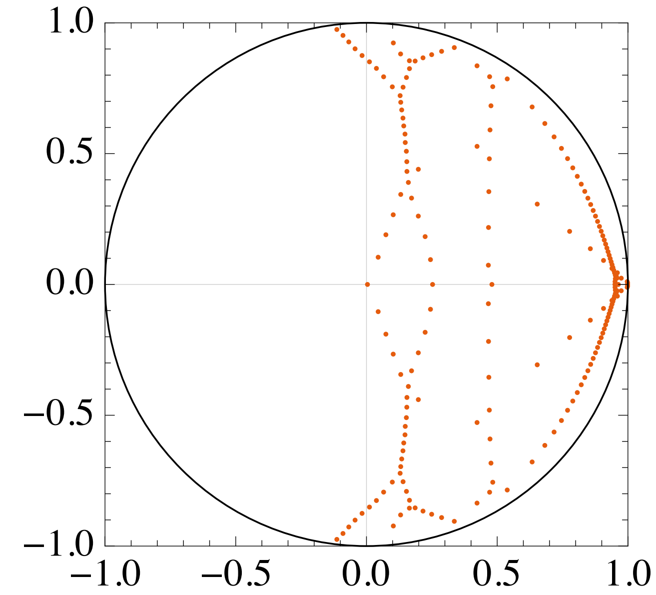
$$J^{\Psi} \cdot (\rho_{n+1,i} - \rho_{n,i}) = \rho_{n,i} - \Phi(\rho_n)_i$$

- When convergence stops, we observe that the Jacobian becomes singular.

E.v.'s of the Newton's method Jacobian at edge of convergence



Bernstein interpolants

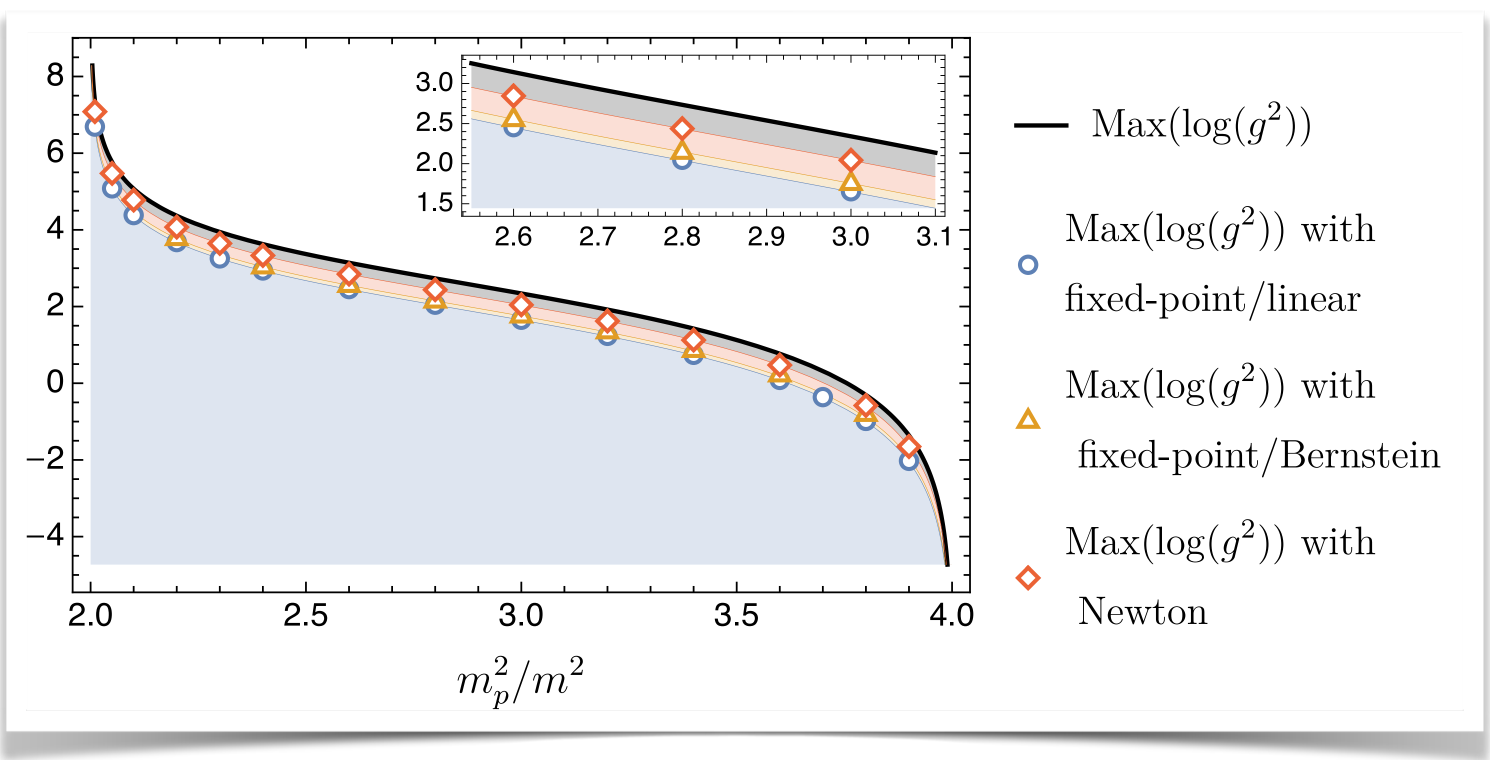


Piecewise-linear interpolants

Results

- Generic convergence in *n*-pole *m*-zero amplitudes. Much better than fixed-point.

- Extend the fixed-point range in 1-pole sector (1-zero), remains a finite strip where divergence.

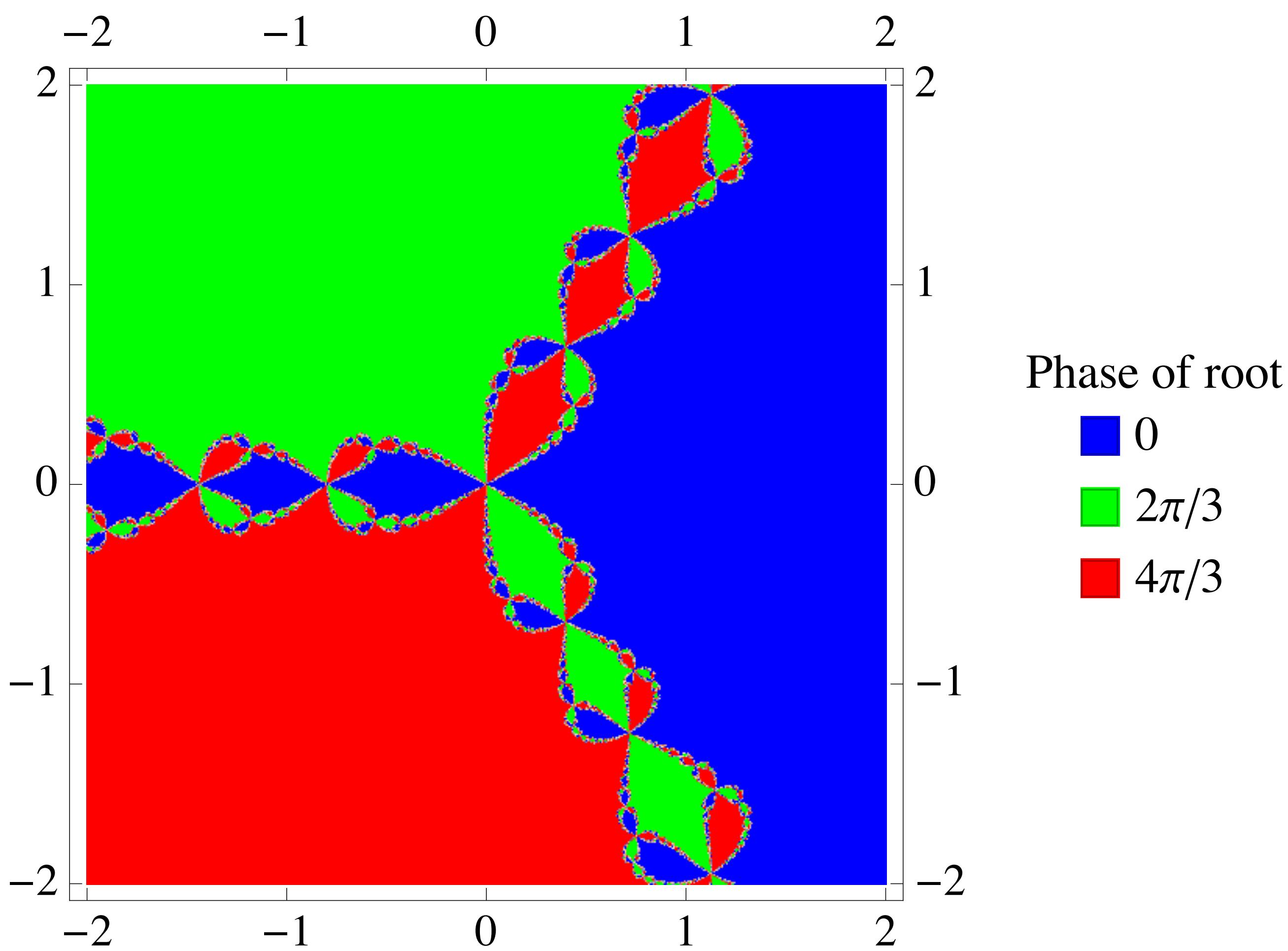


CDD fractal

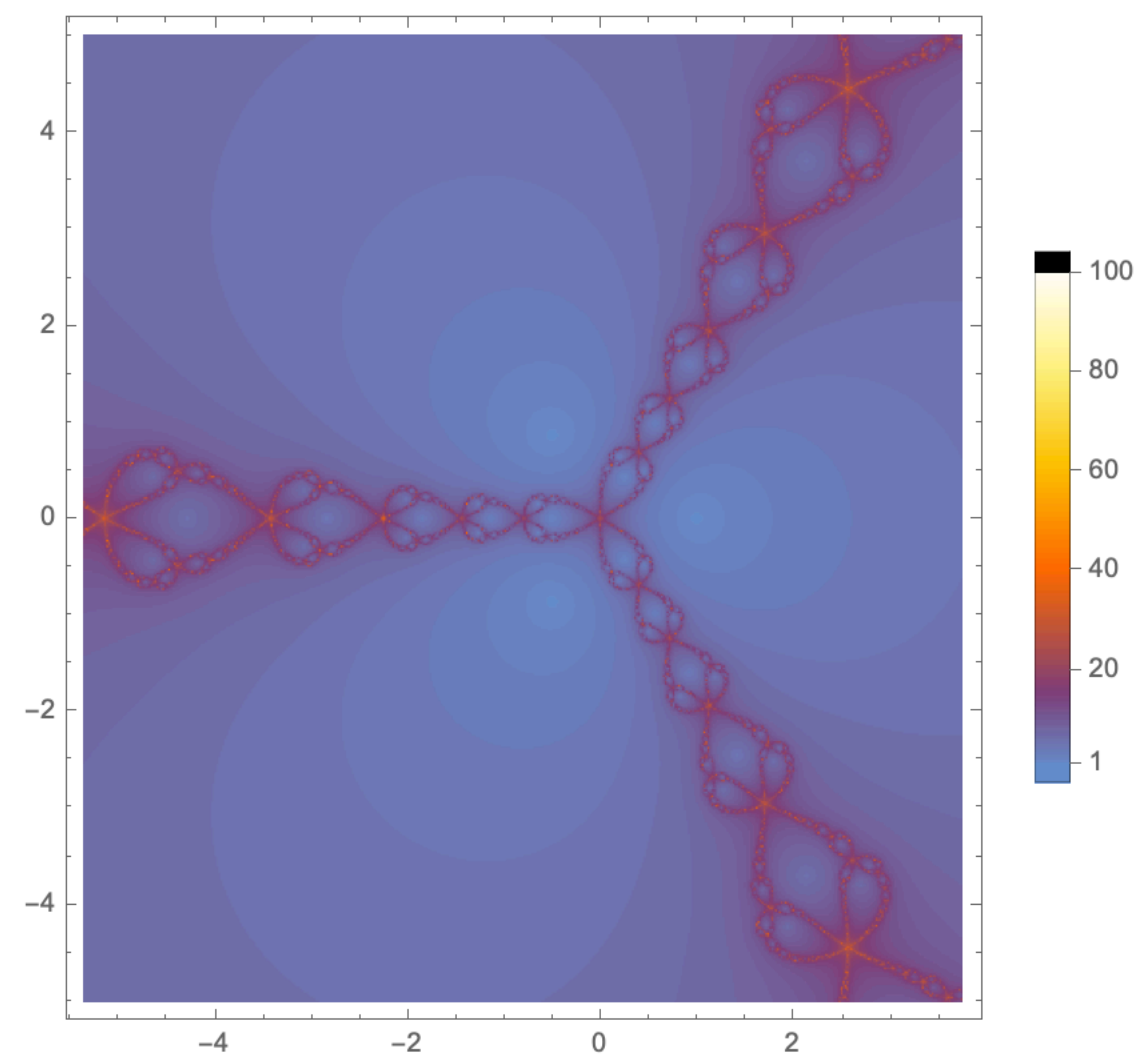
Extended convergence

- Newton's method converges on 1-pole n-zero sectors
- Given c_∞, many solutions are possible, distinguished by position of zeros
- How can the algorithm know what to converge to ?
- Depends on the starting point !

Fractals in 1d




```
P[z_] := z^3 - 1;
JuliaSetPlot[z - P[z]/P'[z], z, PlotLegends → Automatic]
```



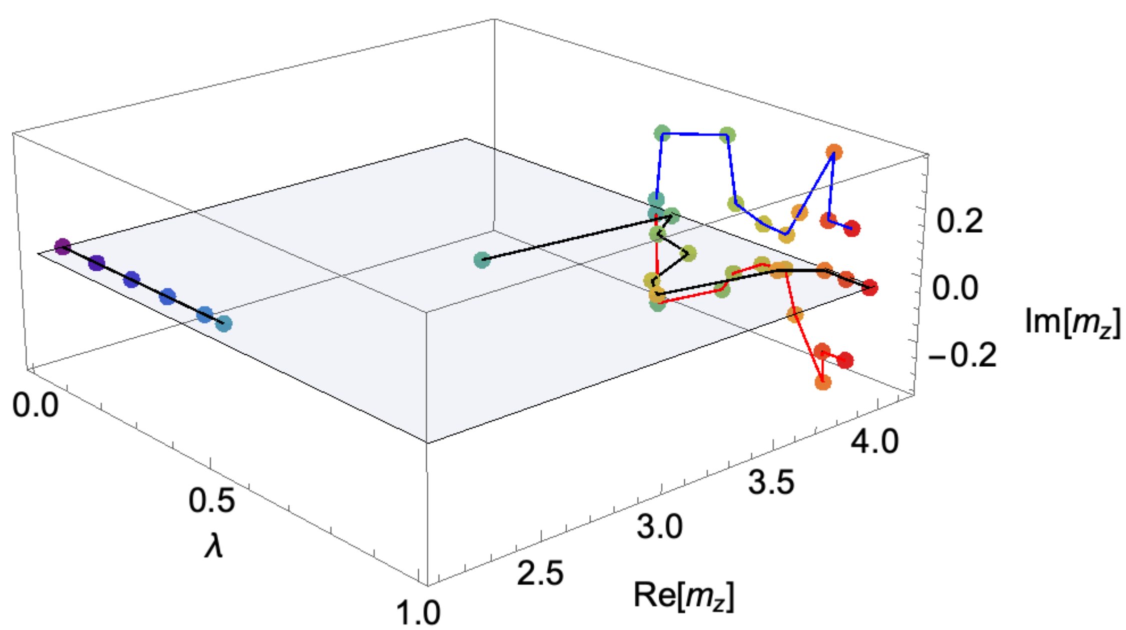
Starting point in between different equally admissible CDD solutions

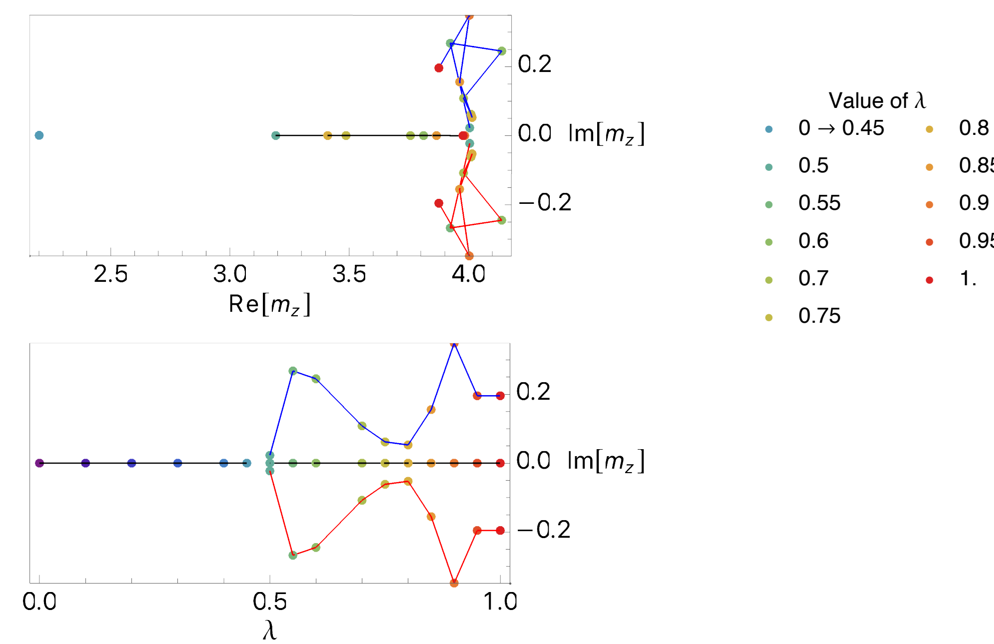
$$\sqrt{m_{z_1}^2(4-m_{z_1}^2)}+\sqrt{m_{z_2}^2(4-m_{z_2}^2)}+\sqrt{m_{z_3}^2(4-m_{z_3}^2)}=\sqrt{m_z^2(4-m_z^2)}$$

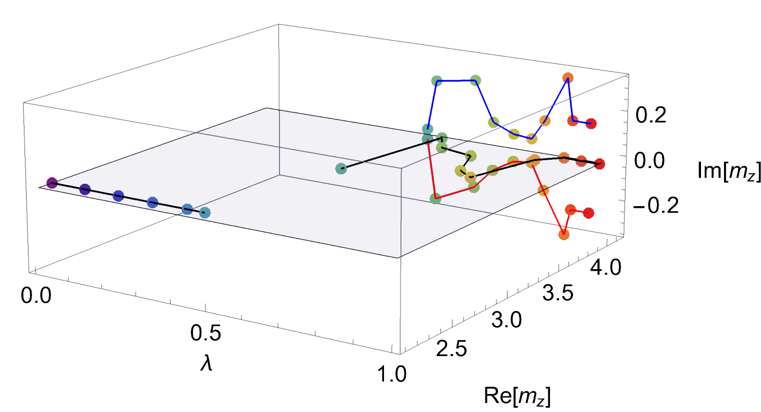
$$f_\lambda(x)=(1-\lambda)\Im T_{1-\text{zero}}(x)+\lambda\Im T_{3-\text{zero}}(x),\quad \lambda\in[0;1]$$

Starting point in between different equally admissible CDD solutions

$$f_\lambda(x) = (1-\lambda)\mathfrak{I}T_{1-\text{zero}}(x) + \lambda\mathfrak{I}T_{3-\text{zero}}(x), \quad \lambda \in [0;1]$$

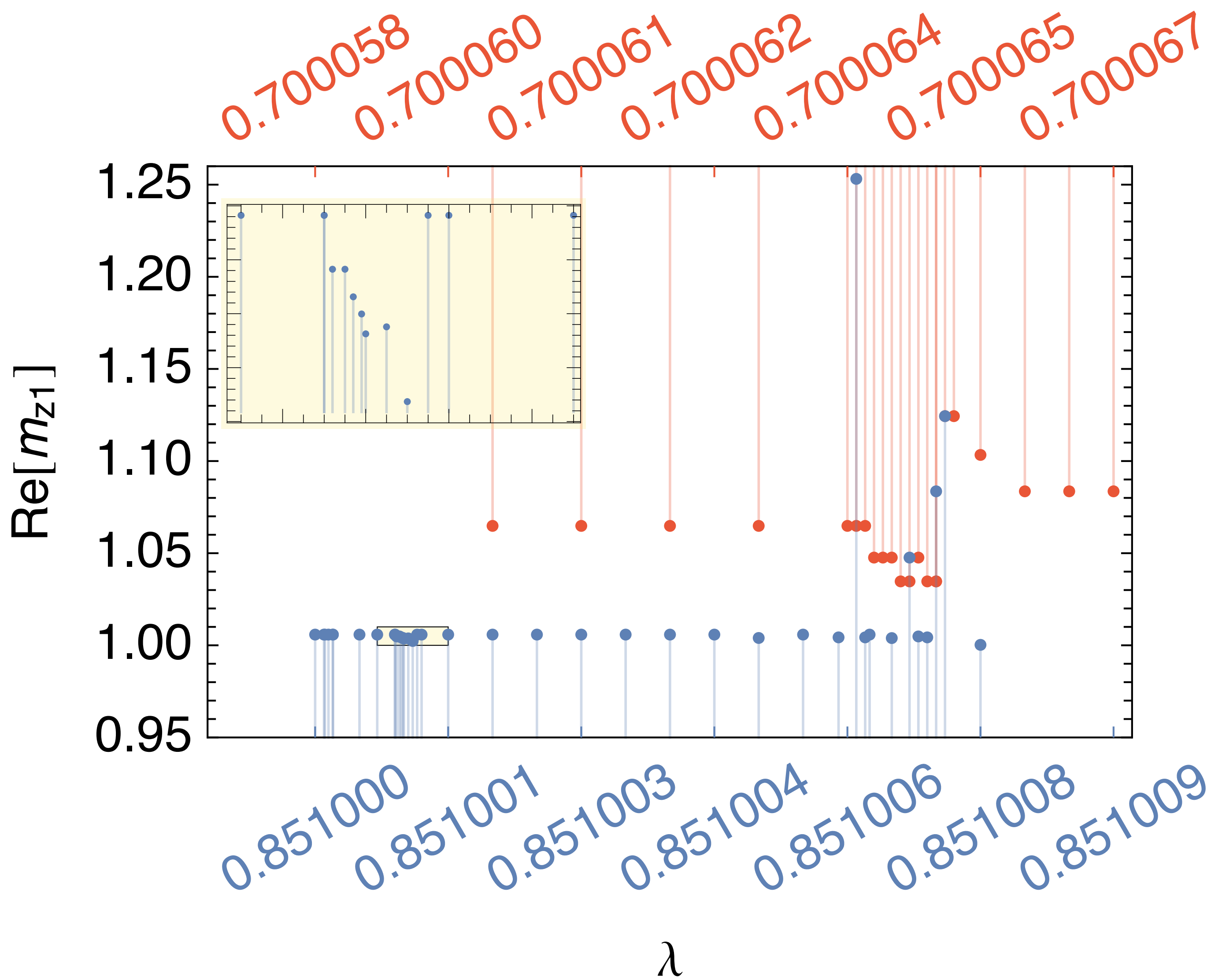




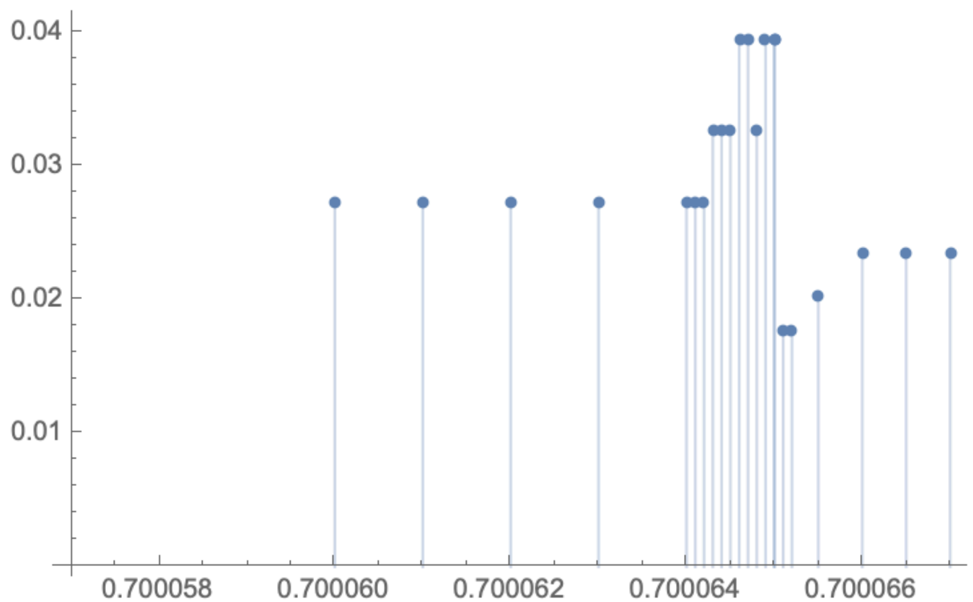


Romanesco broccoli fractal

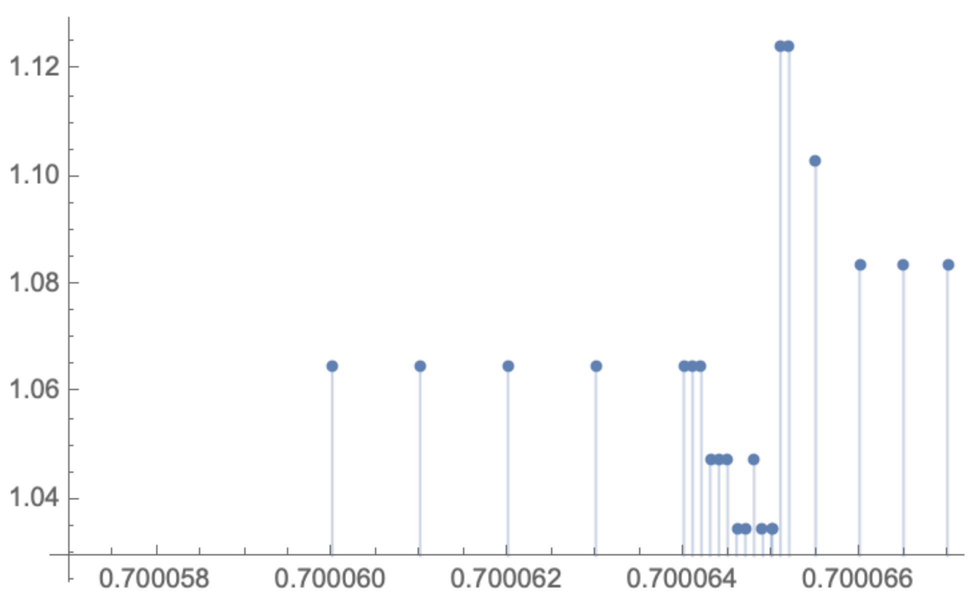
Position of first of the three zeros

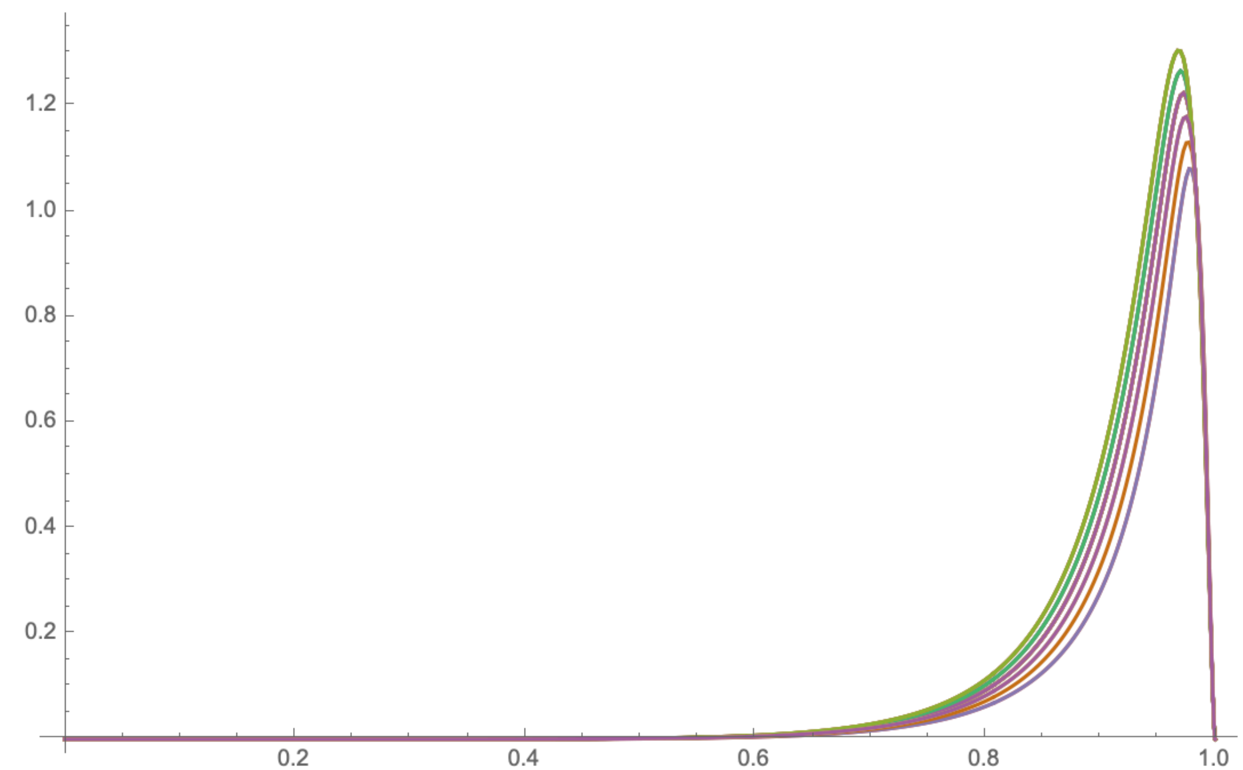


Im[z_2]


0.04
0.03
0.02
0.01
0.700058
0.700060
0.700062
0.700064
0.700066

Re[z_1]


1.12
1.10
1.08
1.06
1.04
0.700058
0.700060
0.700062
0.700064
0.700066


1.2
1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1.0

recap

So, what have we learned ? Why are we doing this ? Where is this going ?

- Atkinson’s program can be made to work.
- It’s the only proposal on the market to implement elastic unitarity.
- We’ve tested it on the simpler case of 2d S-matrices, so as to see how to proceed in 4d.

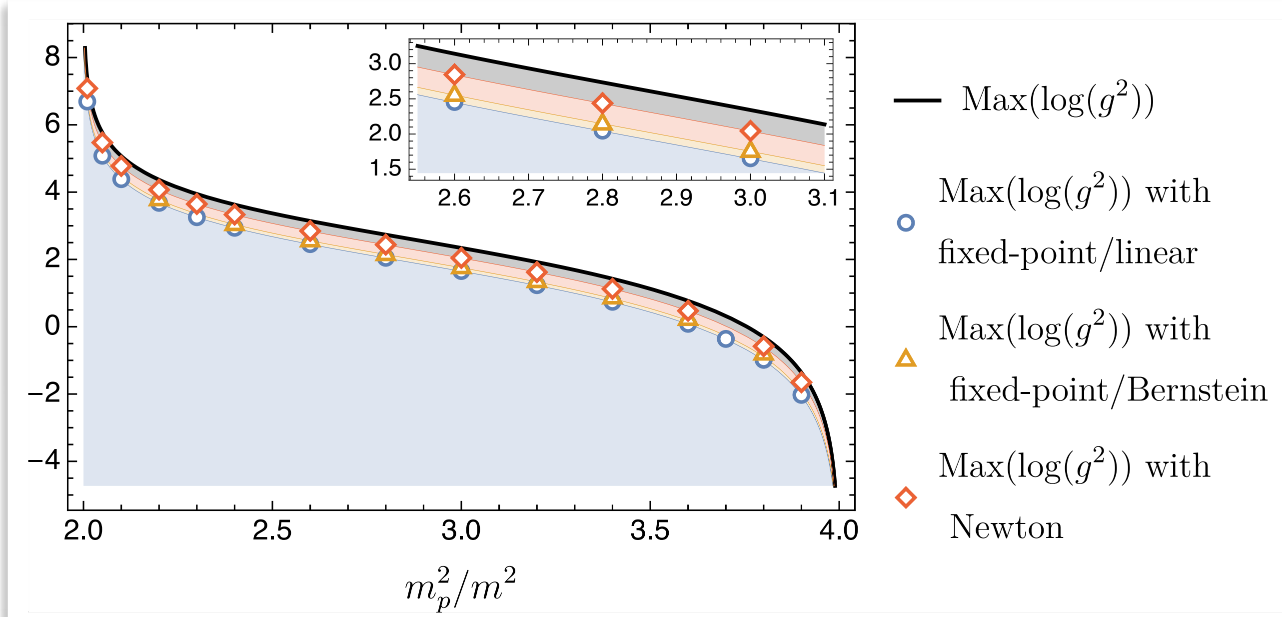
Summary

Summary

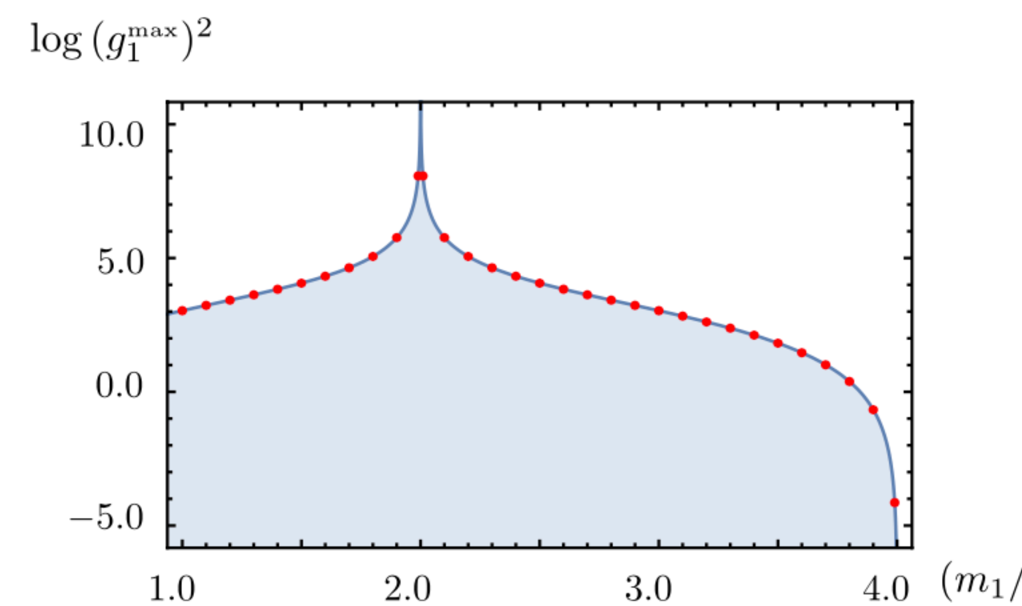
$$\rho_{n+1,i} = \Phi(\rho_{n,j})_i := \frac{x_i}{8\sqrt{1-x_i}}\left(\rho_{n_i}^2 + (G_{ij}\cdot\rho_{n,j} + q_i)^2\right) + v_{inel}(x_i)$$

Essential for speed to be able to compute the matrix G_{ij} in advance

(if you want to construct only one amplitude, that may not be necessary, but to play games and explore function space, speed is necessary)

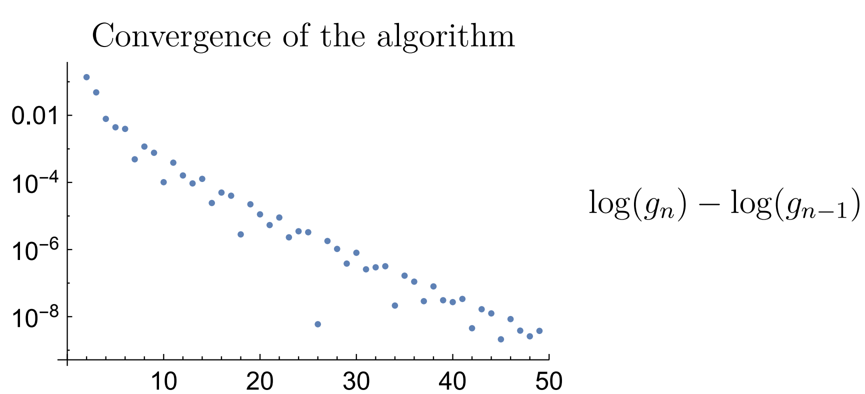


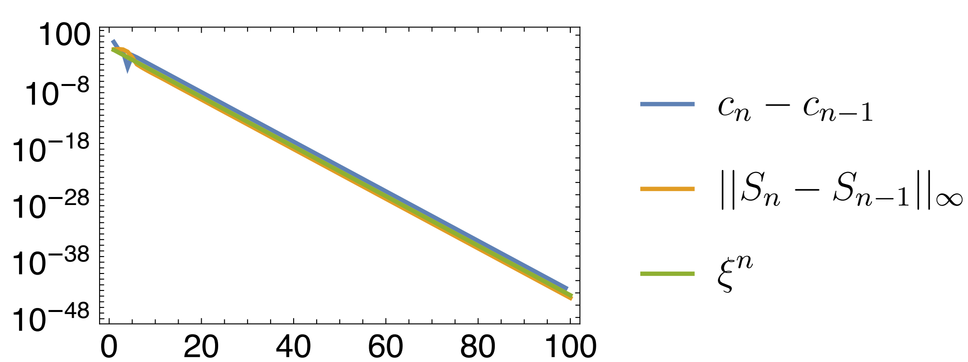
compare with PPTvRV ’16:

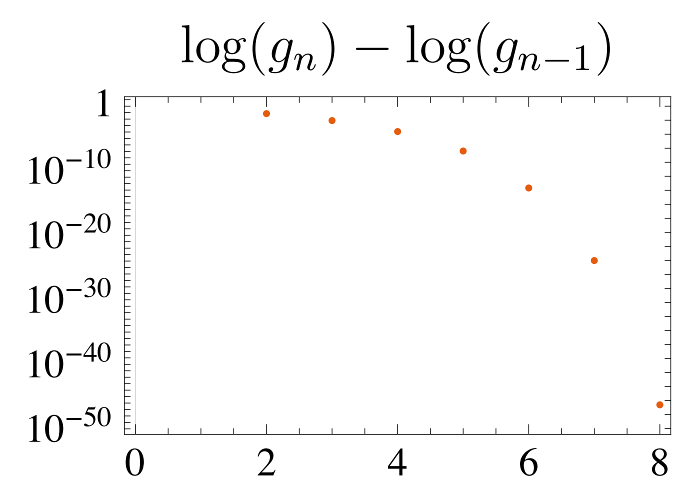


Summary

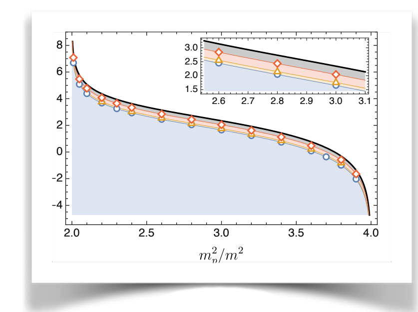
Fixed-Point	**Newton**
$\rho_* = \Phi[\rho_*]$	$\Psi[\rho_*] = \rho_* - \Phi[\rho_*] = 0$
linear ≲ Berstein	linear ≳ Berstein
<u>"linear" convergence</u>	<u>"quadratic" convergence</u>







Open questions in 2d after this work



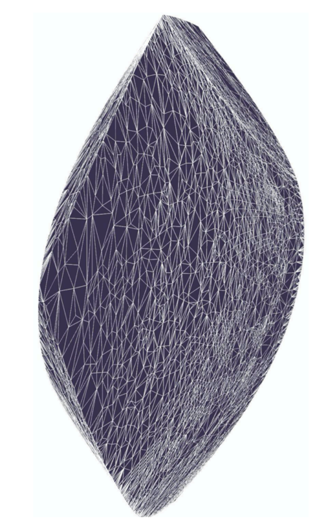
- Implement a method that can deal with singular Jacobians in Newton and fill the gray band
- Can one do a proof *à la* Atkinson's here and how does it compare to our spectral radius analysis ?
- what else can you do with this formalism ? Other 2 to 2 S-matrices ?

Discussion,
Future directions

Add flavours

- In d=2 : add flavours (analytic solution with inelasticity unknown to our knowledge) and probe the O(N) monolith

[arXiv:1909.06495] JHEP **2004** (2020) 142
The O(N) S-matrix Monolith
L. Cordova, Y. He, M. Kruczenski, P. Vieira



analogue of

$$S(s) = S_{\text{elastic}}(s) e^{\int_{4m^2}^{\infty} \frac{ds'}{2\pi i} \log(1 - f_i(s')) \sqrt{\frac{s(s-4m^2)}{s'(s'-4m^2)}} \left(\frac{1}{s'-s} + \frac{1}{s'-(4m^2-s)} \right)}$$

not known

Relation to perturbative expansion

- What is the biggest difference between this and standard unitarity methods, and why does the map converge ? We know QFTs generically have a divergent loop expansion.

- Answer: all is within the assumed the inelastic input.

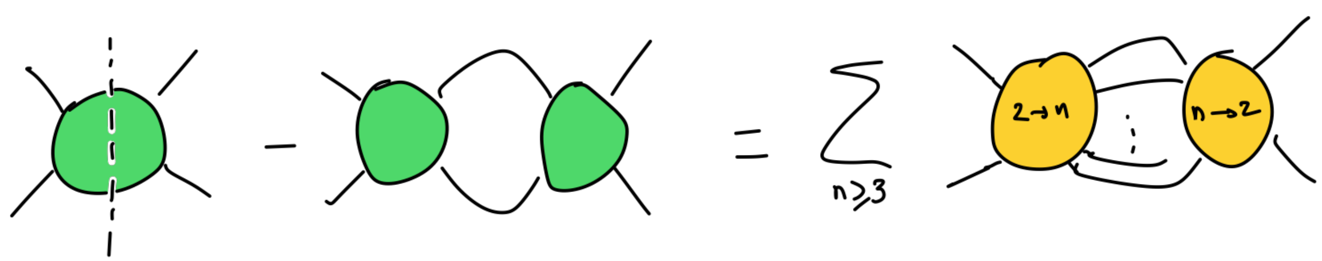
    - if too big so that it mimics pure perturbative expansion, will diverge (cf Gribov's theorem)

    - Iterates much fewer graphs than Feynman graphs so you can resum (similar to eikonal resummation)

- That can be seen as a problem, or as an advantage. We don't have to worry about the issues of summability, and work directly in the space of full S-matrices.

Massless theories

- Can this be used for massless theories ? The separation $\Im T(s) = \dfrac{1}{2\sqrt{s(s-4m^2)}}\,|\,T(s)\,|^2 + v_i(s)$ still holds there, (even though there isn't elastic unitarity)



- In gravity, at high energies, black-holes are produced. Inelastic behaviour might be universal and easy to implement in v_{inel}

Other solvers

- Other numerical solving strategies ? After all we just want to solve a set of coupled non-linear equations. Could a neural-network solve faster and extend again the range of convergence in 2d ?

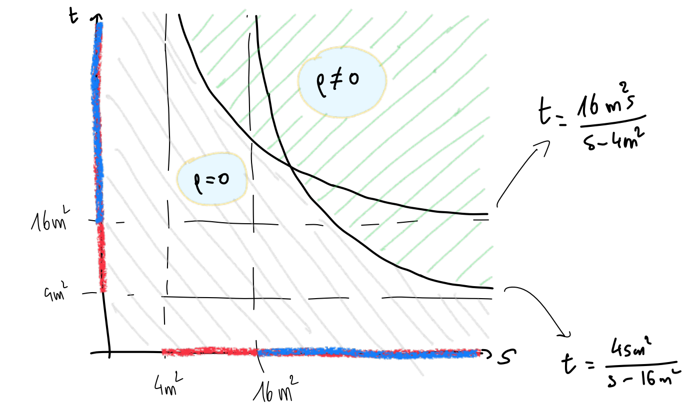
$$\rho_i = \Phi(\rho_j)_i := \frac{x_i}{8\sqrt{1-x_i}}\left(\rho_i^2 + (G_{ij}\cdot\rho_j + q_i)^2\right) + v_{inel}(x_i)$$

Higher dimensions

- Last, but not least: higher dimensions. Challenges:
  - s-grid ⟶ (s,t)-grid; N⟶N^2 points.
  - right-hand side of unitarity equations has a phase space integral. That's an extra N^2 integrals ☹️.
  - Many-layer recursion: for double-discontinuity and single-discontinuity. Add subtractions.
  - Newton's method Jacobian may be hard to compute numerically. Fixed-point will work.

Tourkine, Zhiboedov, work in progress

Higher dimensions



- For v_{inel}, two options:
  - no explicit inelasticity, will be generated automatically by recursion + Aks theorem. What kind of amplitudes are those ? Sort of minimal analogues to integrable theories ?
  - add inelasticity: what will it be ? Adapted from experimental data ?
- There shouldn't be CDD ambiguity because of Aks theorem. If there is, it's also very interesting because very new.

thanks for listening !

Atkinson's proof

- Start from the map $\Phi : L \mapsto L$ where L is a Banach space of Hölder continuous functions

- Hölder continuity :
  $\forall x, y \in [0;1], |f(x) - f(y)| \leq k|x-y|^\alpha$
  for $0 < \alpha < 1$ and $k > 0$

- Define open ball $B = \{f \in L, \|f\| \leq b\}$ for some $b > 0$

- If $\Phi[B] \subset B$, Leray-Schauder principle
  $\implies \exists$ fixed point of Φ

- If Φ is *contracting*, i.e.
  $\|\Phi[f_1 - f_2]\| \leq c\|f_1 - f_2\|$, then the solution is also unique in B.

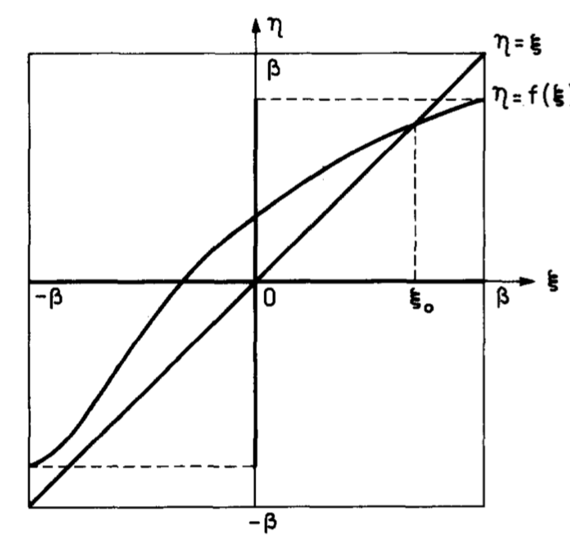


Fig. 1. Illustration of a fixed-point theorem. The image of the interval, $-\beta \leqslant \xi \leqslant \beta$, under the continuous, nonlinear mapping, f, is a subset of the same interval. Therefore the curve $\eta = f(\xi)$ intersects the line $\eta = \xi$ at least once, at a point ξ_0, such that $\xi_0 = f(\xi_0)$.